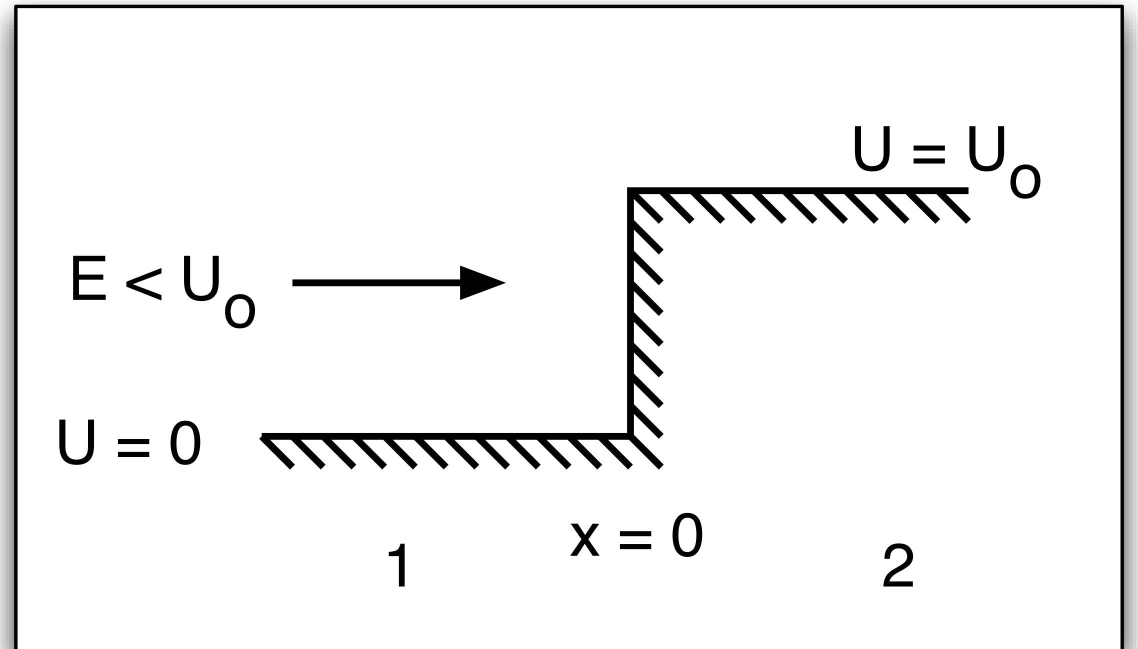
# Electrons incident on an energy step: $E < U_o$




Now consider the case of an electron incident on a step, but with $E < U_o$. Classically, we expect the electron to bounce back from the step.

Spoiler alert: We will find the exact same result quantum mechanically. However, in the process we will observe an interesting twist that will lead to something remarkable if we change the potential configuration slightly.

We use the same approach as in the previous case: find solutions in the two regions and match them at the interface using the connection rules.

We already know the solutions in region 1, where U = 0. This is identical to the previous case:

$$\psi_1(x) = A\exp(ik_1 x) + B\exp(-ik_1 x) \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

Things are a bit more interesting in region 2.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_2(x)}{\partial x^2} + U_o\psi_2(x) = E\psi_2(x)$$

Since $U_o > E$, we can re-write this equation

$$\frac{\partial^2\psi_2(x)}{\partial x^2} - \frac{2m}{\hbar^2}(U_o - E)\psi_2(x) = 0$$

$$\frac{\partial^2\psi_2(x)}{\partial x^2} - \alpha_2^2\psi_2(x) = 0 \qquad \alpha_2 > 0$$

The general solution to this form of the S.E. is

$$\psi_2(x) = C\exp(\alpha_2 x) + D\exp(-\alpha_2 x)$$

The two terms represent growing and decaying exponentials. These are not traveling waves. They do still have the oscillatory time-dependence that we've seen for time-independent potentials, but they are simply exponential terms with no traveling component. These exponential functions are known as evanescent waves.

Now we need to match the solutions at the boundary, but first a physical argument: the first term above, Cexp( $\alpha_2$x), represents a growing exponential. As x increases, this term will "blow up". We know that this cannot be an allowable wave function. So we will preclude the impossibility by removing the term from the solution (i.e. we'll set C to 0).

$$\psi_1(0) = \psi_2(0) \qquad\qquad \psi_1'(0) = \psi_2'(0)$$

$$A + B = D \qquad\qquad ik_1 A - ik_1 B = -\alpha_2 D$$

As done previously, we can define a reflection amplitude

$$\frac{B}{A} = -\frac{\alpha_2 + ik_1}{\alpha_2 - ik_1}$$

The reflection coefficient is

$$R = \frac{j_R}{j_I} = \frac{k_1}{k_1}\frac{|B|^2}{|A|^2}$$

$$= \frac{\alpha_2^2 + k_1^2}{\alpha_2^2 + k_1^2} = 1$$

It is guaranteed that the electron will be reflected. This is probably not surprising. There may be a phase shift during reflection, but there is no doubt that the electron will bounce back.

Of much more interest is what is happening on the other side. The wave function is:

$$\psi_2(x) = D\exp(-\alpha_2 x)$$

The probability density in region 2 is

$$|\psi_2(x)|^2 = |D|^2 \exp(-2\alpha_2 x)$$

This says that there is a non-zero probability that the electron can be on the other side of the barrier. The probability drops off exponentially into the barrier, so the electron can't penetrate far, but it can penetrate. This is completely non-classical.

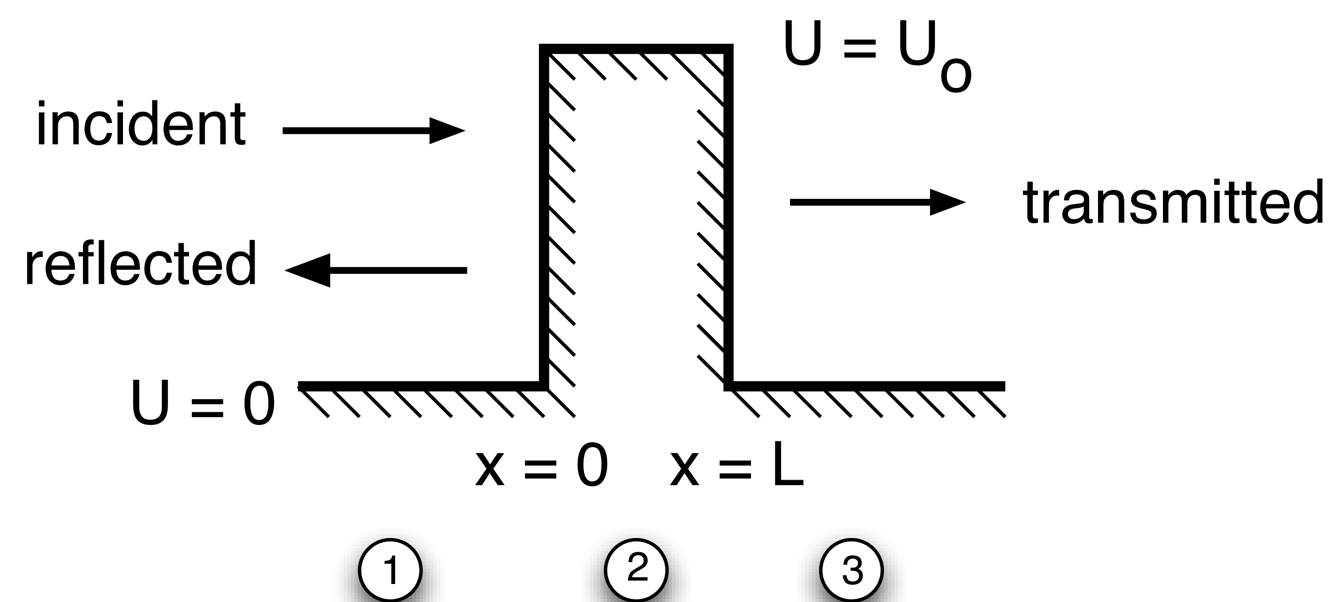
The average depth of penetration into the barrier is $2\alpha_2$.

If the barrier were very thin, the electron may be able to squeeze past it and appear on the other side.

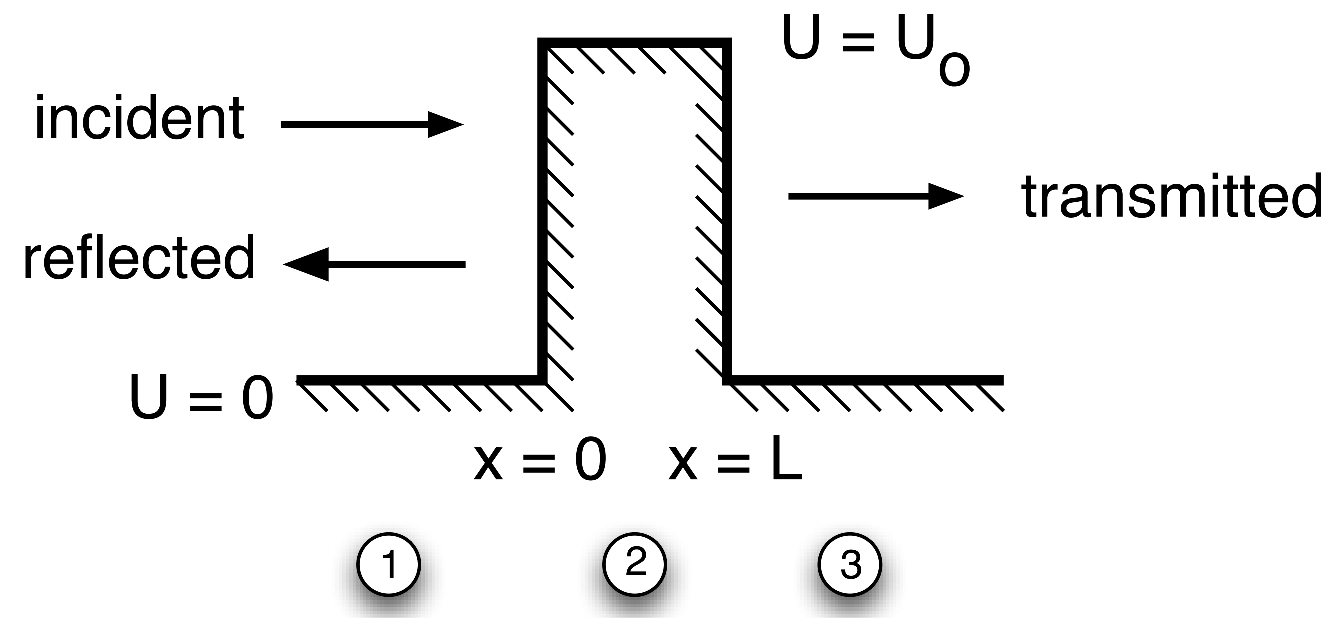
# Tunneling through a square barrier

So can an electron penetrate a classically impenetrable barrier? It's easy to set up a potential to look at this problem. The math is slightly tedious, but the result makes the effort worthwhile.

A simple square barrier is shown at right. The barrier has energy height of $U_o$ and width of L. An electron is incident from the left, and it's energy is less than the barrier height, $E < U_o$.



We take the same approach as previously, but now there are three regions to consider. The general solutions in each region are easy.




1-Free electron incident from the left with a likely reflection.

$$\psi_1(x) = A\exp(ik_1x) + B\exp(-ik_1x)$$

2-With E < U$_o$, growing and decaying exponentials. (Note that we can't arbitrarily throw out the growing exponential, since the barrier width is finite in this case.)

$$\psi_2(x) = C\exp(\alpha_2x) + D\exp(-\alpha_2x)$$

3-Wave that has tunneled through — outgoing only. ($k_3 = k_1$)

$$\psi_3(x) = F\exp\left[ik_1(x-L)\right]$$

Since there shouldn't be any confusion about where k and α come from, we will leave off the subscripts.

$$\psi_1(x) = A\exp(ikx) + B\exp(-ikx)$$
$$\psi_2(x) = C\exp(\alpha x) + D\exp(-\alpha x)$$
$$\psi_3(x) = F\exp\left[ik(x-L)\right]$$

Now use connection rules at x=0 and x=L.

$$\psi_1(0) = \psi_2(0) \qquad A + B = C + D$$
$$\psi_1'(0) = \psi_2'(0) \qquad ikA - ikB = \alpha C - \alpha D$$

$$\psi_2(L) = \psi_3(L) \qquad C\exp(\alpha L) + D\exp(-\alpha L) = F$$
$$\psi_2'(L) = \psi_3'(L) \qquad \alpha C\exp(\alpha L) - \alpha D\exp(-\alpha L) = ikF$$

It looks like 4 equations in 5 unknowns. Of course, the goal is to find a transmission amplitude ratio, F/A. So we use the four equations to eliminate B, C, and D.

The algebra is a bit tedious, but after grinding through it (Make sure that you can do it yourself.)

$$\frac{F}{A} = \frac{i4\alpha k}{\exp(-\alpha L)\,(\alpha + ik)^2 - \exp(\alpha L)\,(\alpha - ik)^2}$$

Yikes! We can simplify this somewhat by realizing that in many barrier problems it is likely that

$$\exp(\alpha L) >> \exp(-\alpha L)$$

In which case, the transmission expression simplifies to

$$\frac{F}{A} \approx \frac{(-i4\alpha k)\exp(-\alpha L)}{(\alpha - ik)^2}$$

The tunneling coefficient is then

$$T = \frac{j_T}{j_I} = \frac{k_3|F|^2}{k_1|A|^2} = \frac{|F|^2}{|A|^2} = \frac{(4\alpha k)^2 \exp(-2\alpha L)}{(k^2 + \alpha^2)^2}$$

$$T = \frac{(4\alpha k)^2 \exp(-2\alpha L)}{(k^2 + \alpha^2)^2}$$

This can be put into a more user-friendly form by expressing k and $\alpha$ in terms of energies.

$$T = \frac{16E\,(U_o - E)}{U_o^2} \exp(-2\alpha L)$$

The pre-factor varies between 0 and 4, depending on the value of E and $U_o$. However, the important part of the expression is the exponential dependence on the quantity $\alpha L$. The tunneling probability goes down rapidly as $\alpha L$ increases. Obviously, this happens as the barrier gets wider or becomes higher (which causes $\alpha$ to increase).

The "full-blown" (i.e. non-approximate) tunneling expression is

$$T = \frac{1}{1 + \left[\frac{U_o^2}{4E(U_o - E)}\right] \sinh^2(\alpha L)}$$

You should show that this expression reduces to the approximate form when $\alpha L$ is greater than about 3. The algebra required to obtain this expression can be viewed on the class web site

Tunneling shows up in all kinds of places. Perhaps one of the most common is in the operation of flash memory, where electrons tunnel from the channel of a MOSFET to the "floating" gate, where they become trapped. The extra charge changes the FET's I-V characteristics, thus "storing" information in the FET.

# Example

An electron (mass = 9.11x10–31 kg) with energy of 1.0 eV is incident on an energy barrier of height 1.5 eV. The barrier is 0.5 nm thick. Calculate the probability that the electron tunnels through.

With the given energies, the pre-factor is:

$$\frac{16E\left(U_o - E\right)}{U_o^2} = \frac{16\left(1eV\right)\left(1.5eV - 1eV\right)}{\left(1.5eV\right)2} = 3.56$$

The exponential factor is (be careful with units here)

$$\alpha L = \sqrt{\frac{2m\left(U_0 - E\right)}{\hbar^2}}L$$

$$= \sqrt{\frac{2\left(9.11 \times 10^{-31}kg\right)\left(1.5eV - 1eV\right)\left(1.6 \times 10^{-19}J/eV\right)}{\left(1.055 \times 10^{-34}J \cdot s\right)^2}}\left(0.5 \times 10^{-9}m\right) = 1.81$$

$$T = 3.56\exp\left[-2\left(1.81\right)\right] = 0.0955$$

# Example 2

Repeat example 1, for the following conditions: $E$ = 0.25 eV and $L$ = 0.5 nm, $E$ = 1.3 eV and $L$ = 0.5 nm, $E$ = 1 eV and $L$ = 2 nm, and $E$ = 1 eV and $L$ = 0.25 nm.

The calculation is exactly the same as example 1.

$E$ = 0.25 eV and $L$ = 0.5 nm → $T$ = 0.00727

$E$ = 1.3 eV and $L$ = 0.5 nm → $T$ = 0.1875

$E$ = 1 eV and $L$ = 2 nm → $T = 1.83 \times 10^{-6}$

$E$ = 1 eV and $L$ = 2 nm → $T$ = 0.583

With the probability so high, we should be suspicious of the answer. See the next problem.

# Example 3

What if the electron from example 1 were lighter than a "normal" electron? For example, electrons in GaAs have an effective mass of $0.063m_o$, where $m_o$ is the rest mass of a free electron. What would be the tunneling probability if the electron were as light as one in GaAs?

$0.063m_o = 5.74\text{x}10^{-32}$ kg. Using that, along with the same numbers from example in the tunneling formula:

$$E = 1 \text{ eV},\ L = 0.5 \text{ nm, and } m = 0.063m_o \rightarrow T = 1.43$$

What? The problem here is that the tunneling probability is so high that the approximate formula from slide 10 doesn't apply. (Using the exact formula gives T =0.801).

The important point here is that light electrons tunnel more easily and so we might observe tunneling effects more readily in semiconductors like GaAs. We will see more about this later.