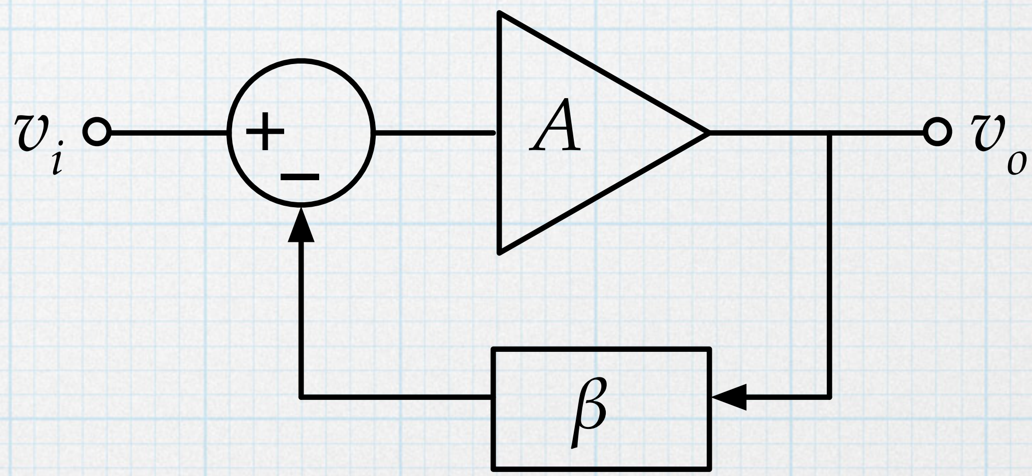
Stability

In a feedback system, an interesting condition can occur, leading to a potential problem and also a potential opportunity.

We have seen numerous examples where the feedback network is frequency dependent, and we have just examined the gain-bandwidth limit of the open-loop amp. Both A and β can contribute to the frequency dependence of the closed-loop gain.

$$G(s) = \frac{A(s)}{1 + A(s)\beta(s)}$$



Focusing on sinusoidal behavior, $s = j\omega$:

$$G(j\omega) = \frac{A(j\omega)}{1 + A(j\omega)\cdot\beta(j\omega)}$$

It is certainly conceivable that there might be a frequency where

$$A\left(j\omega\right)\beta\left(j\omega\right) = -1.$$

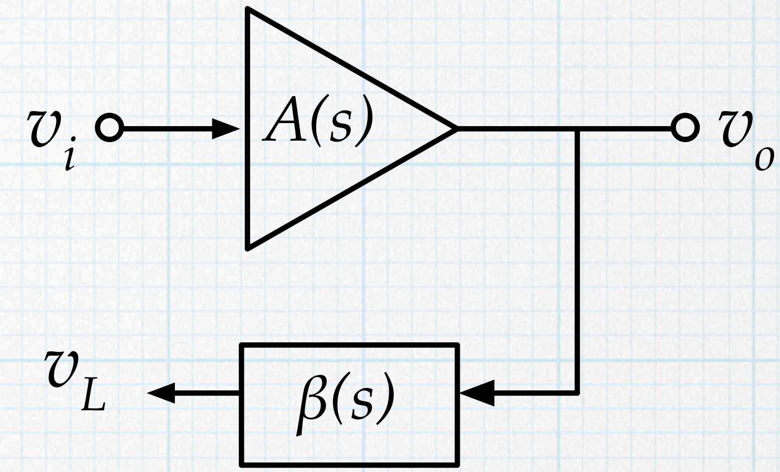
In that case, the closed-loop gain would tend towards infinity! The closed-loop amplifier would not be stable.

This is not quite as easy as the condition above implies, since there are requirements for both magnitude and phase.

$$|A\beta| = 1 \text{ and } \theta = 180^\circ,$$

but the possibility certainly exists.

The loop gain



The key to understanding stability is the loop gain, $L(s) = V_L(s)/V_i(s)$.

$$L(j\omega) = A(j\omega) \cdot \beta(j\omega)$$

When considering issues of stability, we generally are concerned with the properties of the amp and usually the feedback circuit will not have have frequency dependence — so assume that β is a constant.

$$L(j\omega) = \beta \cdot A(j\omega)$$

The "natural" frequency of amp is low-pass — the operation of the transistors degrades as the frequency increases, giving the amp an overall low-pass characteristic. Writing the loop gain in magnitude and phase form:

$$L(j\omega) = \beta \left| A(j\omega) \right| \exp\left[j\theta(\omega) \right]$$

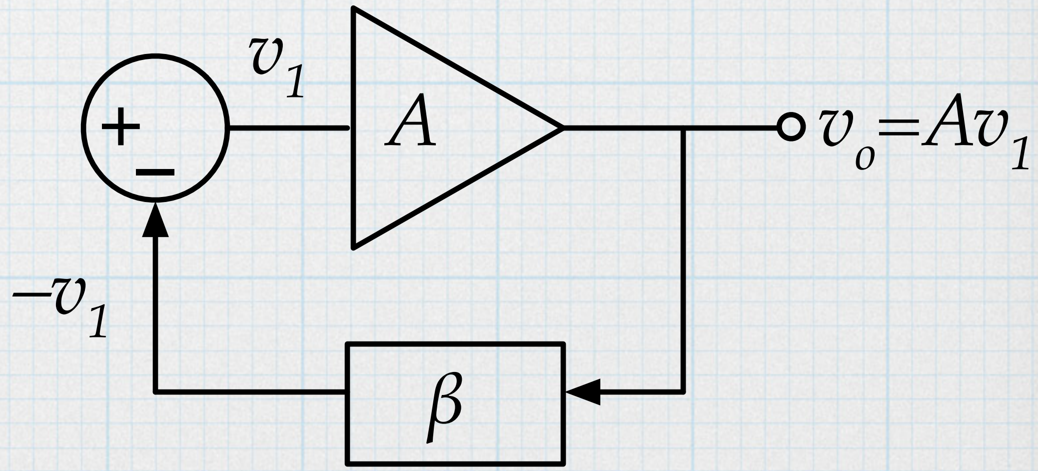
If, at some frequency, the phase shift is 180° (denote this frequency as ω_{180}), then

$$L\left(j\omega_{180}\right) = -\beta\left|A\left(j\omega_{180}\right)\right|$$

At the frequency ω_{180}, what was intended to negative feedback has become positive feedback! Everything now depends on the magnitude at that frequency. If the magnitude is less than one, the closed loop gain will actually become *bigger* than the open-loop gain. This a bit weird, but it we are OK.

But if the magnitude happens to be exactly 1 at ω_{180}, then we are in trouble. In the case the loop gain is exactly –1, and the closed-loop gain goes to infinity. This is the definition of an unstable circuit. An infinitely large gain implies that the circuit can have a finite output with no input!

As the signal with frequency ω_{180} propagates around loop, it will sustain itself! There is no input, but the output will be a sinusoid at that very specific frequency. Thus, an unstable circuit will oscillate.



Furthermore, if the circuit has this instability in the loop frequency response, it is guaranteed to oscillate. Any disturbance in a circuit (a bit of noise, a switch closing, etc) will induce a voltage somewhere along the path of the circuit. That disturbance will probably have a component at $\omega = \omega_{180}$. That component will be sustained by the positive feedback at that frequency.

And having the magnitude be bigger than 1 at $\omega = \omega_{180}$ does not help. In that case, the circuit will still be unstable, but the oscillations will now grow exponentially with time. The growth will be limited only by the power supply limits of amplifier.

Note that, in order to have the loop gain reach the point where the phase = –180°, the transfer function of the amp must third order or higher. A first-order low-pass will have a maximum phase shift of –90° — it will never become unstable. A second-order low-pass has maximum phase shift of –180°, but this only occurs as $\omega \rightarrow \infty$, so the oscillation frequency would be "extremely high". So, in order for an amp to become unstable, it's low-pass response must be third-order or higher. Unless precautions are taken, most have amps will have high-order transfer functions and stability is a concern.

Stability – another view

Another way to think about the stability is to consider what can happen to a loop gain when the feedback loop is closed. As an example, consider an amp that has a third-order transfer function of

$$A(s) = \frac{108}{(s+1)\left(s+2+j2\right)\left(s+2-j2\right)} = \frac{108}{s^3 + 5s^2 + 12s + 8}$$

If this is combined with a feedback loop with $\beta = 0.25$, then

$$L(s) = \beta A(s) = \frac{26}{s^3 + 5s^2 + 12s + 8}$$

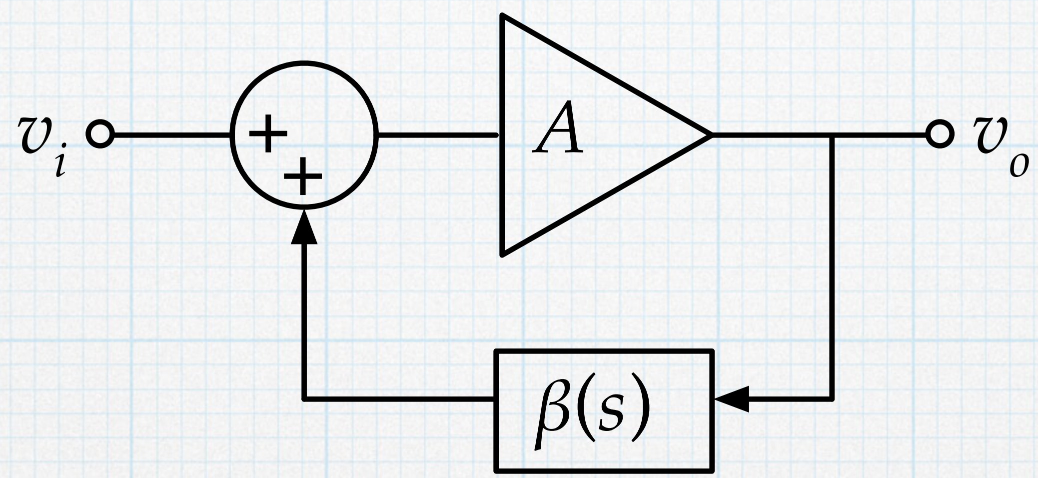
There is nothing wrong or unstable with this function. However, when the feedback loop is closed, the resulting function is

$$G(s) = \frac{A(s)}{1 + A(s)\beta} = \frac{\frac{108}{s^3 + 5s^2 + 12s + 8}}{1 + \frac{26}{s^3 + 5s^2 + 12s + 8}} = \frac{-108}{s^3 + 5s^2 + 12s - 18}$$

The closed-loop function has a different denominator, with different roots!

Linear oscillators

We can turn the idea of trying to make amplifiers stable on its head by taking a nominally stable amplifier and adding a feedback circuit that will cause the closed-loop system to become unstable. We use *positive feedback* – a sample of the output feed back and *added* to the input.



The feedback network has a very specific frequency dependence, $\beta \rightarrow \beta(s)$. With positive feedback, the closed-loop transfer function is

$$G(s) = \frac{A}{1 - A\beta(s)}$$

If the frequency-dependence of the feedback circuit relies on an LC resonance, it is usually referred to as a "tank circuit".

To create the instability, the denominator of the closed-loop transfer function must be zero.

$$1 - A\beta\left(s\right) = 0$$

Which is to say that the loop gain, $A\beta(s)$, must be equal to 1.

$$A\beta\left(s\right) = 1$$

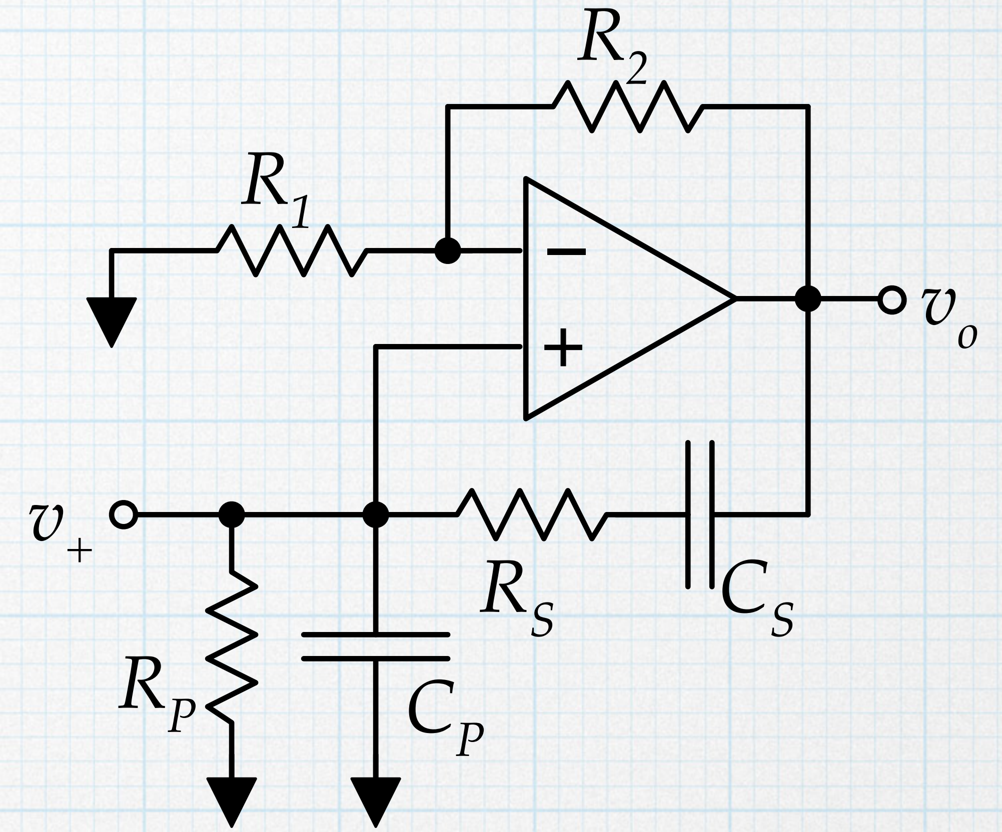
$$A\beta\left(j\omega\right) = 1e^{j0^\circ}$$ This is known as the *Barkhausen criterion*.

Another way of describing what is happening is that the poles of the transfer function must occur in the right-half plane. Ideally, the poles would be right on the imaginary axis, meaning that the denominator has zeros at $s = \pm j\omega_o$. Thus $1 - A\beta(s)$ would be of the form $s^2 + \omega_o^2$.

In practice, it is very difficult to design a circuit that has poles exactly on the imaginary axis. Generally, the circuit is designed to have poles that are slightly in the right-half plane. Then the oscillations will grow exponentially with time and some sort of amplitude control will be needed to keep the oscillations close to sinusoidal.

The Wien-bridge circuit.

A simple and popular oscillator circuit is based on the Wein bridge. The circuit is essentially a non-inverting amp with a frequency-dependent voltage divider connecting the output back to the non-inverting input.



While not a requirement, the circuit is usually designed with $R_S = R_P$ and $C_S = C_P$.

The loop gain is easily calculated.

$$v_o = \left(1 + \frac{R_2}{R_1}\right) v_+$$

$$v_+ = \frac{Z_p}{Z_p + Z_s} v_+ = \frac{\frac{R_p}{1 + sR_pC_p}}{\frac{R_p}{1 + sR_pC_p} + R_s + \frac{1}{sC_s}} = \frac{1}{1 + \frac{R_s}{R_p} + \frac{C_p}{C_s} + sR_sC_p + \frac{1}{sR_pC_s}} v_o$$

Combining the two and doing a bit of familiar algebraic re-arrangement, we see the the the loop function has the familiar bandpass form.

$$L(s) = \left(1 + \frac{R_2}{R_1}\right) \cdot \frac{s\left(\frac{1}{R_s C_p}\right)}{s^2 + s\left(\frac{1}{R_s C_s} + \frac{1}{R_p C_p} + \frac{1}{R_s C_p}\right) + \frac{1}{R_s R_p C_s C_p}}$$

Often, the circuit is designed with $R_s = R_p$ and $C_s = C_p$ (although this is not a requirement).

$$L(s) = \left(1 + \frac{R_2}{R_1}\right) \cdot \frac{s\left(\frac{1}{RC}\right)}{s^2 + s\left(\frac{3}{RC}\right) + \frac{1}{(RC)^2}}$$

substituting $s = j\omega$,

$$L(j\omega) = \left(1 + \frac{R_2}{R_1}\right) \cdot \frac{j\omega\left(\frac{1}{RC}\right)}{\left[\frac{1}{(RC)^2} - \omega^2\right] + j\omega\left(\frac{3}{RC}\right)}$$

When the feedback loop is closed, the circuit will oscillate at the frequency where the phase angle of the loop function is zero.

$$\theta = 90^\circ - \arctan\left[\frac{\dfrac{3\omega_{osc}}{RC}}{\dfrac{1}{(RC)^2} - \omega_{osc}^2}\right] = 0^\circ$$

To meet the requirement, the arctangent term must go to 90°, which happens when the denominator goes to infinity.

$$\omega_{osc} = \frac{1}{RC}$$

Next, the magnitude must be equal to 1 at the oscillation frequency.

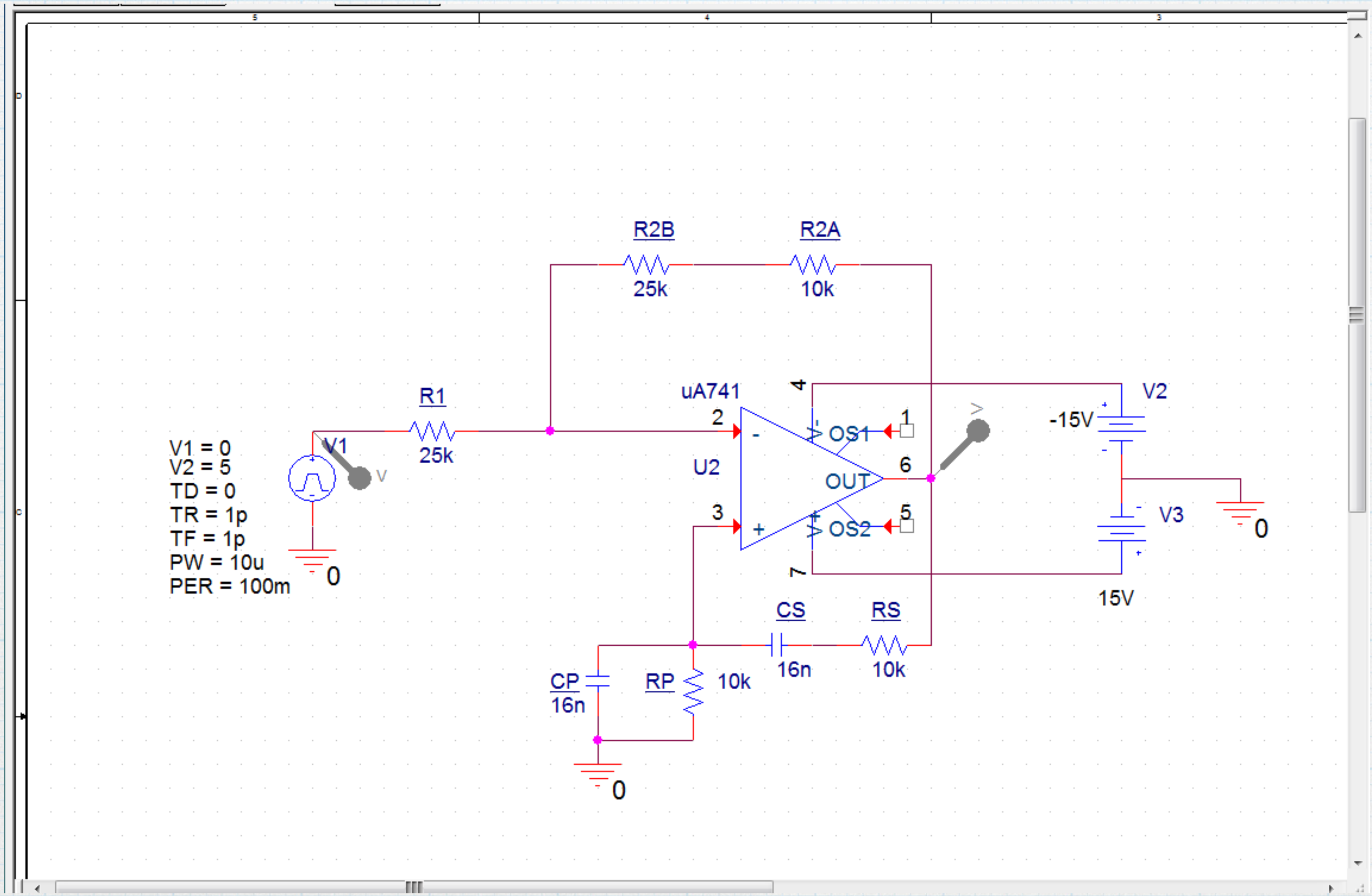
$$\left| L\left(j\omega_{osc}\right) \right| = \frac{1 + \dfrac{R_2}{R_1}}{3} = 1$$

With the result that $1 + R_2/R_1 = 3$ or $R_2/R_1 = 2$.

Designing a simple Wien-bridge oscillator is straight-forward — choose RC to set the desired frequency and then adjust the gain to start the oscillation.

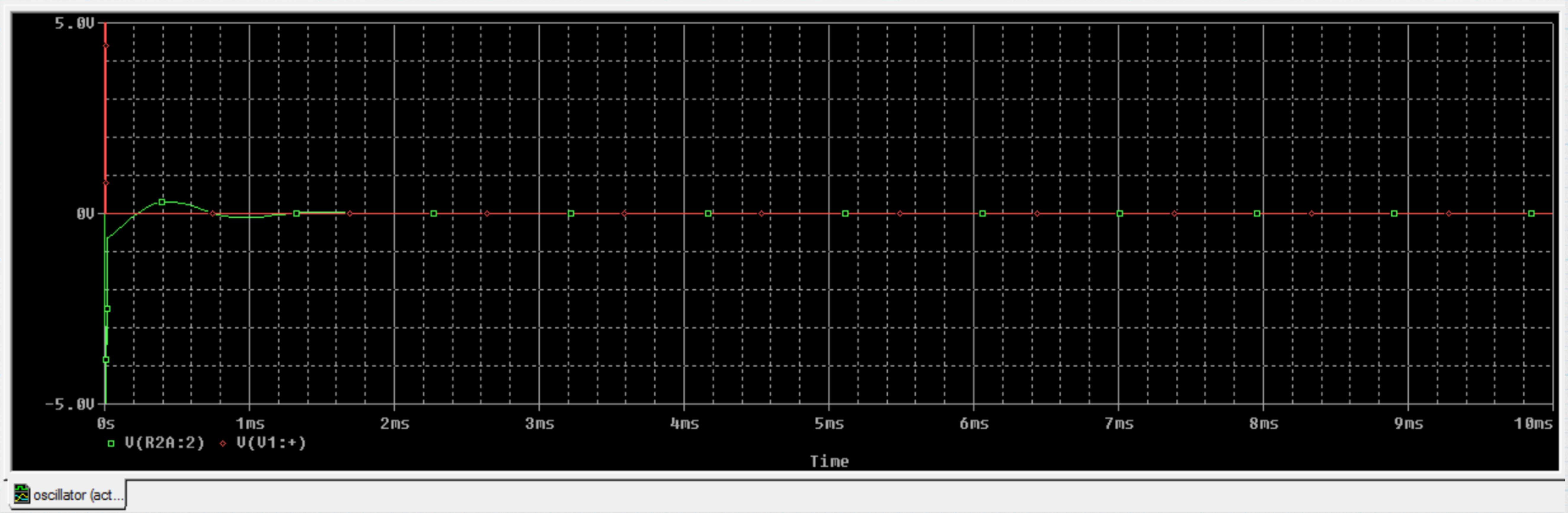
A PSPICE example.

Use the 741 op amp model in PSPICE. $R_P = R_S = 10\ \mathrm{k\Omega}$. $C_P = C_S = 16\ \mathrm{nF}$. The expected oscillation frequency is 1 kHz.

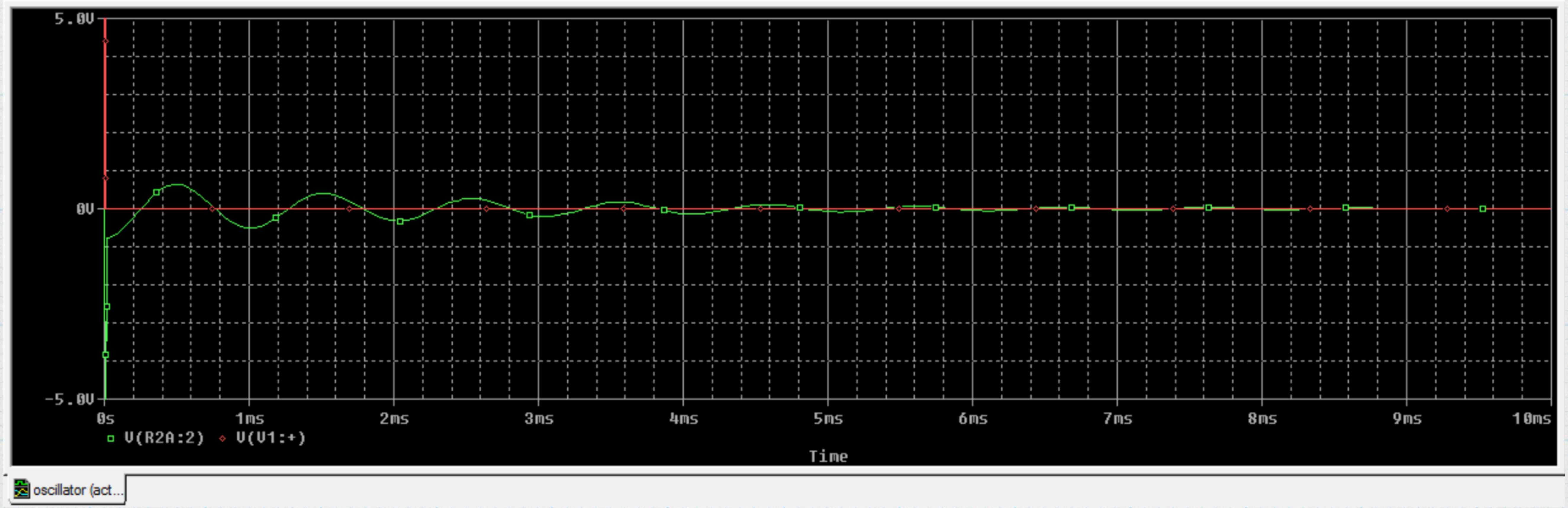


SPICE needs some kind of transient to "kick-start" the oscillation. Use a single short pulse at the input to get things started. Use transient analysis to see voltage as a function of time.

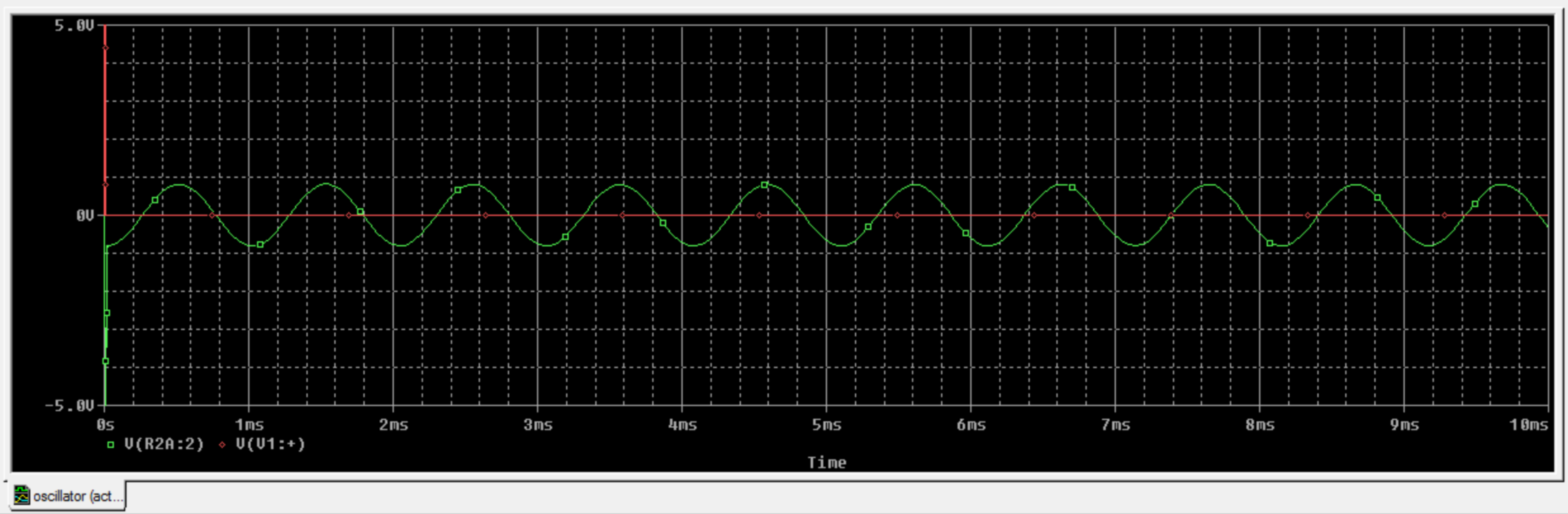
With R_2 = 35 kΩ and R_1 = 25 kΩ, there is not enough gain to start the oscillation. $(1+ R_2/R_1) = 2.4$.



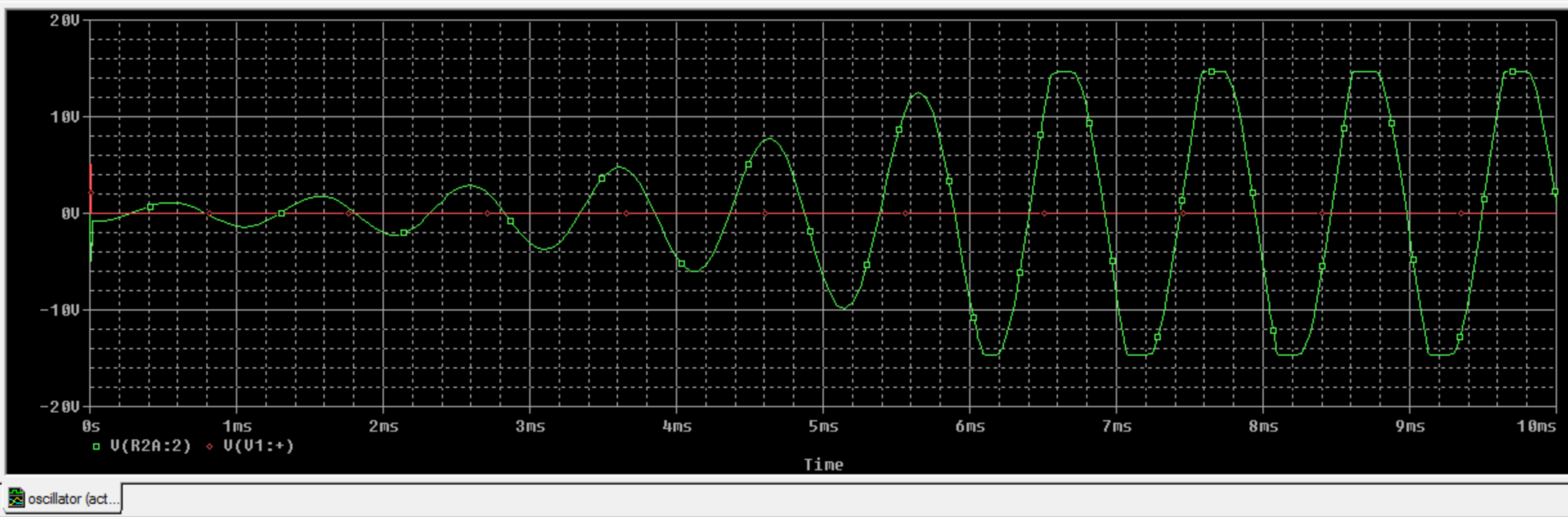
Increasing the gain to 2.86 (R_2 = 39 kΩ and R_1 = 21 kΩ, gives a few more wiggles, but the gain is still too small to sustain the oscillations.



With R_2 = 40 kΩ and R_1 = 20 kΩ, the gain is exactly 3 and the circuit oscillates with a clean sine wave.

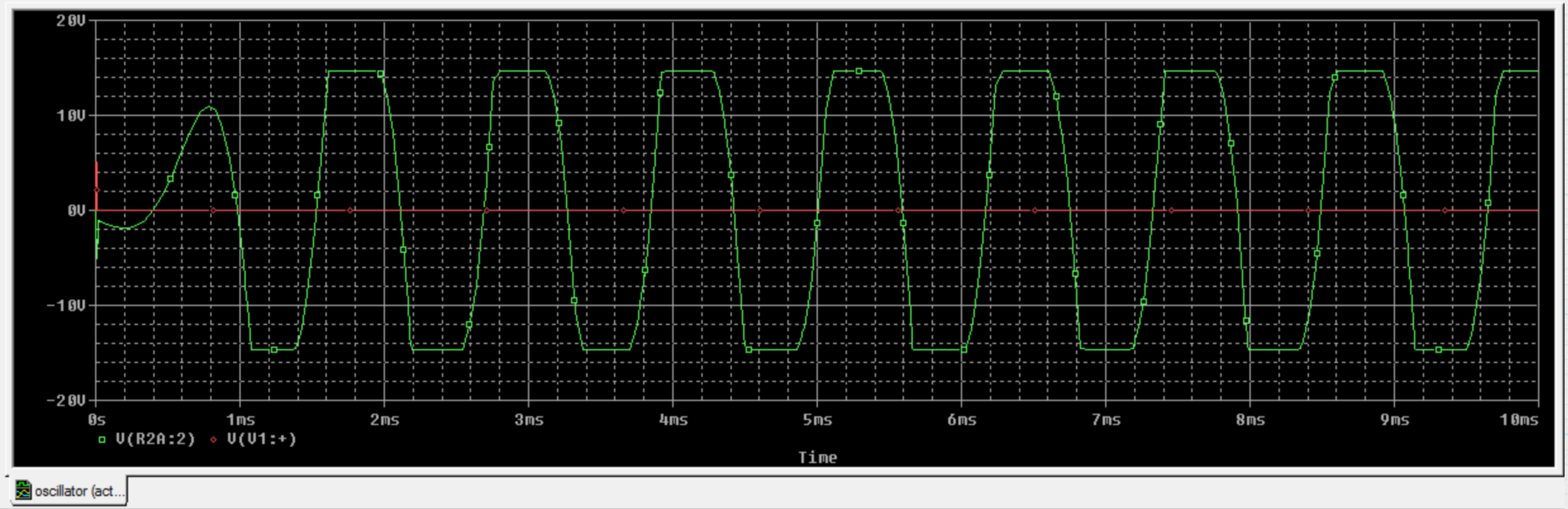


With R_2 = 41 kΩ and R_1 = 19 kΩ, the gain is 3.16 and the oscillations grow with time until clipped by the power supply limits.



The poles of the transfer function are now in the right-half plane, meaning that this is an unstable exponential growth in the response.

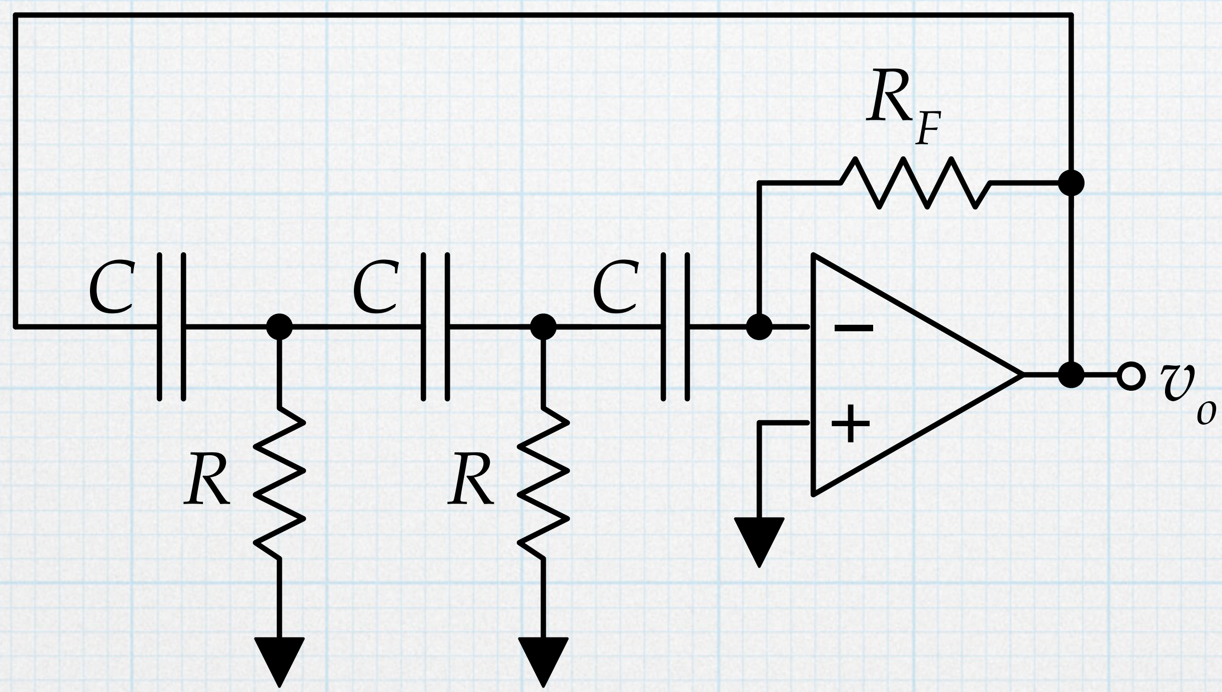
Increasing the gain even more (R_2 = 35 kΩ and R_1 = 15 kΩ giving a gain of 3.33) causes the oscillations to grow – and clip – even faster.



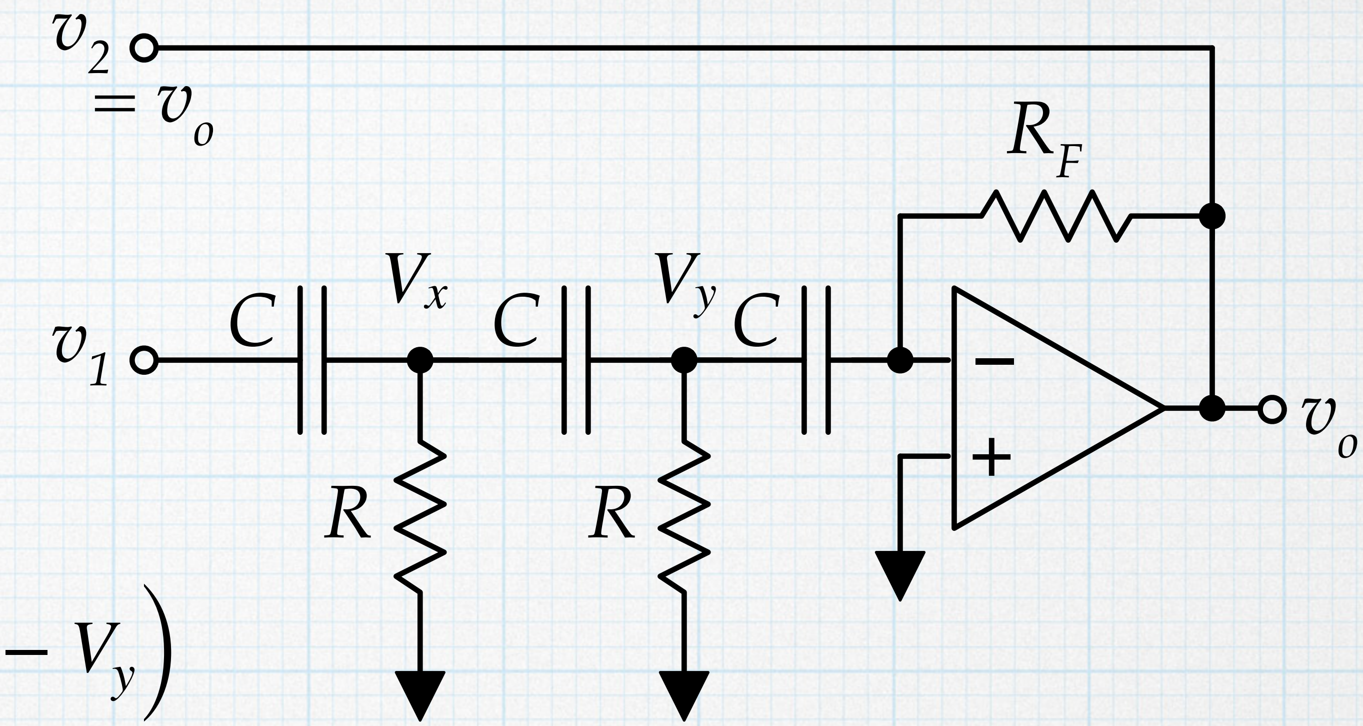
The poles are even farther into the right-half plane, making the exponential growth that much stronger.

phase-shift oscillator

A phase shift oscillator uses an inverting amp for the gain, which comes with its own 180° phase shift. So the tank circuit must provide another 180° of phase shift to get back to 0° (360°) to meet the Barkhausen criterion. This requires a 3-pole circuit, since a 2-pole circuit will get to 180° only at $f \to \infty$.



Calculate the loop gain. (Break the loop…)



$$sC\left(V_1 - V_x\right) = \frac{V_x}{R} + sC\left(V_x - V_y\right)$$

$$sC\left(V_x - V_y\right) = \frac{V_y}{R} + sC \cdot V_y$$

$$sC \cdot V_y = \frac{-V_o}{R_f}$$

Turning the algebra crank, we can come up with the loop function. (You should work this out for yourself.)

$$L(s) = \frac{V_o}{V_1} = \frac{-s^3 R_f C}{3s^2 + s\left(\frac{4}{RC}\right) + \frac{1}{(RC)^2}}$$

Switching to AC analysis ($s = j\omega$)

$$L(j\omega) = \frac{+j\omega^3 R_f C}{\left[\frac{1}{(RC)^2} - 3\omega^2\right] + j\left(\frac{4\omega}{RC}\right)}$$

Note the sign change.

The phase expression is $\theta = 90° - \arctan\left[\frac{\omega^3 R_f C}{\frac{1}{(RC)^2} - 3\omega^2}\right]$

To make the phase go to zero, the arctan term must be 90°, which occurs when the denominator goes to zero.

$$\omega_{osc} = \frac{1}{\sqrt{3}RC}$$

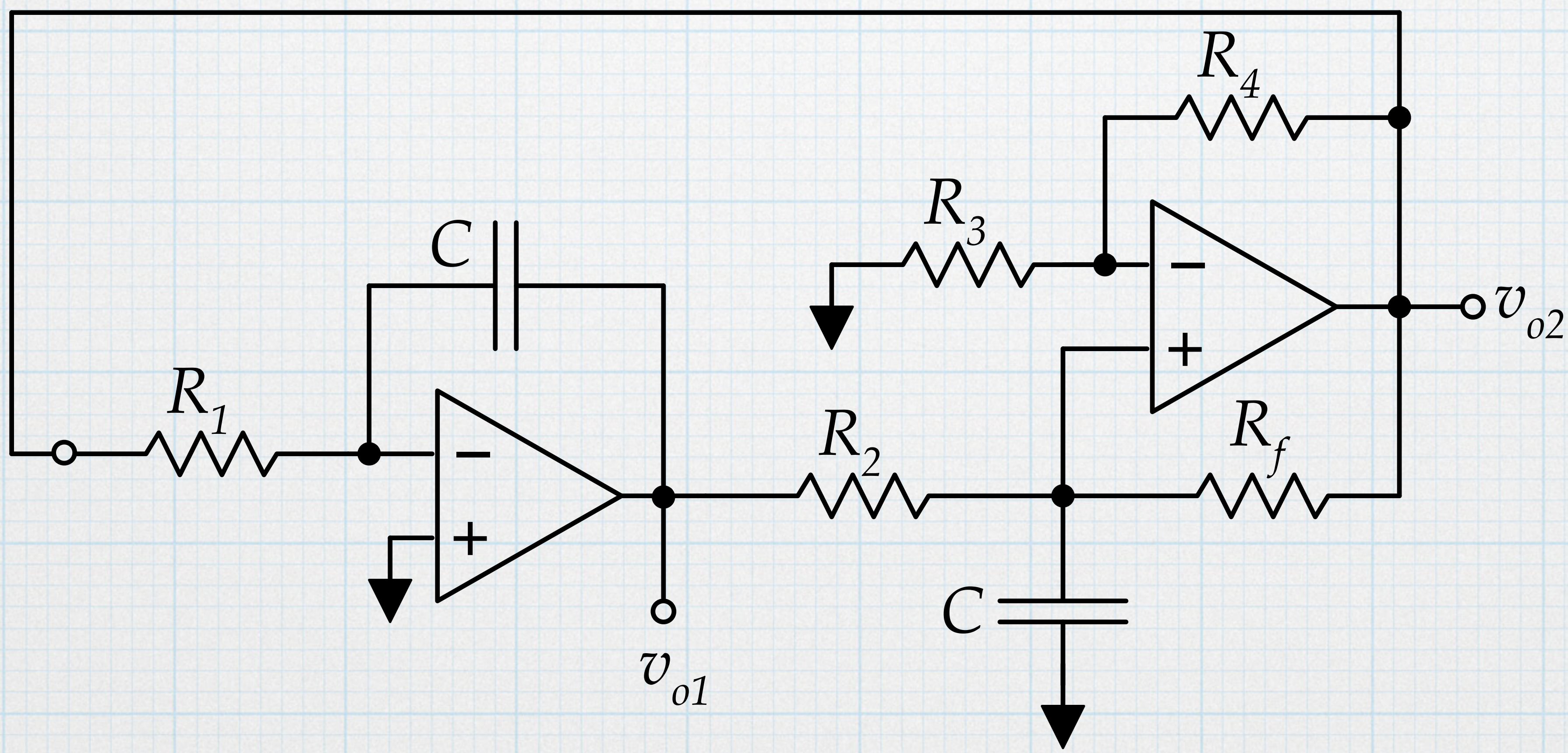
At the oscillation frequency, the magnitude of L must be 1:

$$\left|L(j\omega_{osc})\right| = 1 = \frac{\omega_{osc}^3 R_f C}{\sqrt{\left[\frac{1}{(RC)^2} - 3\omega_{osc}^2\right]^2 + \left(\frac{4\omega_{osc}}{RC}\right)^2}} = \frac{R_f}{12R}$$

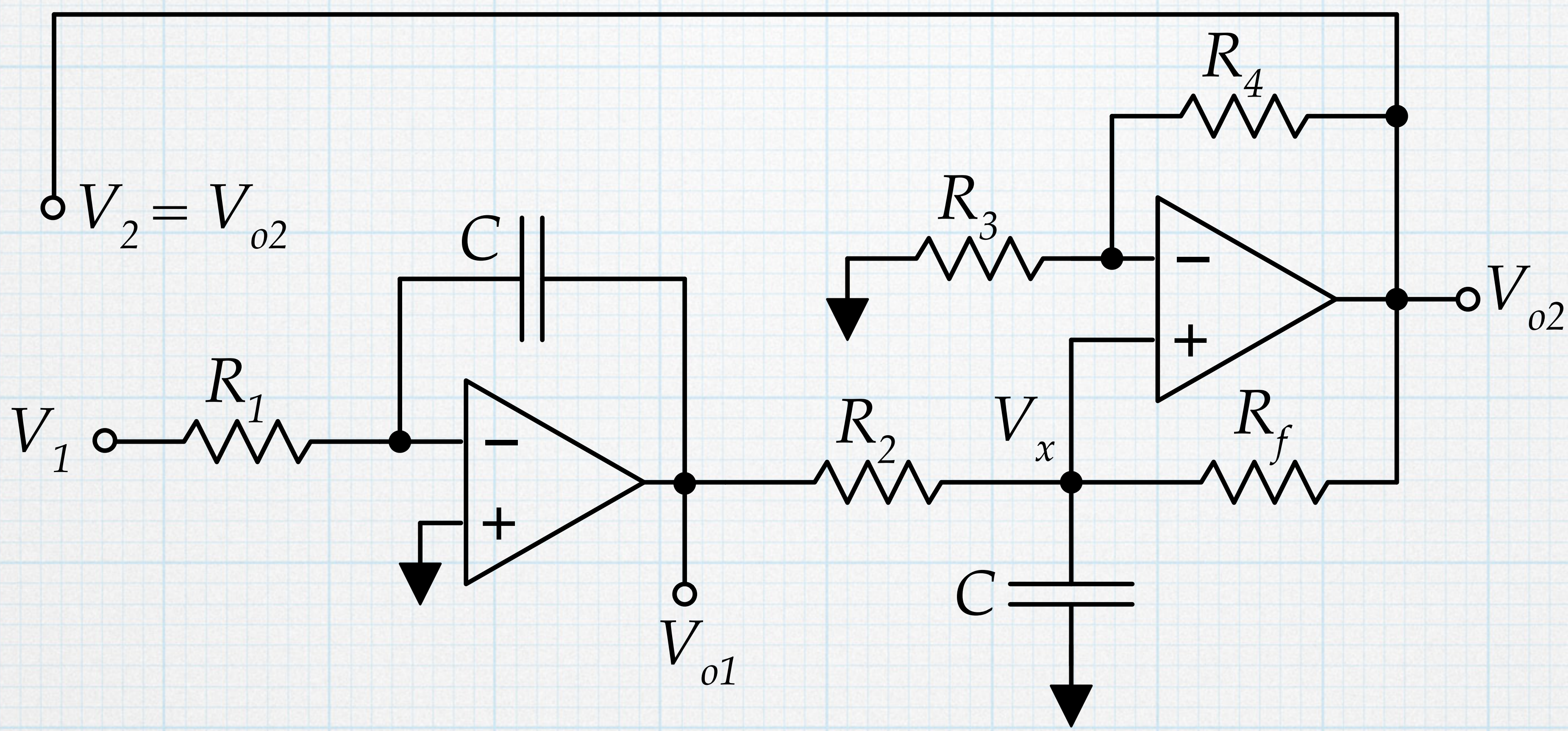
$$R_f = 12R$$

quadrature oscillator

This is an interesting one. It uses an straight integrator circuit followed by a non-inverting amp with a single-pole tank circuit. The output of each op-amp will oscillate sinusoidally, and the two sinusoids are 90° out of phase (i.e. in quadrature). This can be useful in some



Calculate the loop gain. (Break the loop somewhere…)



From the integrator: $V_{o1} = -\dfrac{1}{sR_1C}V_1$

At node x: $\dfrac{V_{o1} - V_x}{R_2} + \dfrac{V_{o2} - V_x}{R_f} = sCV_x$

From the inverting amp: $V_{o2} = \left(1 + \dfrac{R_4}{R_3}\right)V_x$

Putting it all together, we can find the loop gain:

$$L(s) = \frac{V_2}{V_1} = -\frac{1 + \frac{R_4}{R_3}}{s\left[1 - \frac{R_2 R_4}{R_2 R_f}\right] R_1 C + s^2 R_1 R_2 C^2}$$

Switching to AC analysis ($s = j\omega$)

$$L(j\omega) = -\frac{1 + \frac{R_4}{R_3}}{j\omega\left[1 - \frac{R_2 R_4}{R_2 R_f}\right] R_1 C - \omega^2 R_1 R_2 C^2}$$

The phase will be zero (Barkhausen criterion) when

$$\frac{R_2 R_4}{R_2 R_f} = 1$$

When the phase is zero, the magnitude is

$$|L| = \frac{1 + \frac{R_4}{R_3}}{\omega^2 R_1 R_2 C^2}$$

The magnitude needs to be 1 (or bigger) at the oscillation frequency. There are many ways to choose component values to meet the two conditions. One common and simple combination is:

$$R_2 = R_3 = R_4 = R_f = 2R_1$$

In that case, the loop gain reduces to:

$$L(s) = -\frac{1}{s^2 R_1^2 C^2}$$

$$L(j\omega) = \frac{1}{\omega^2 R_1^2 C^2}$$

The oscillation will occur when

$$|L| = \frac{1}{\omega_o^2 R_1^2 C^2} = 1$$

$$\omega_o = \frac{1}{R_1 C}$$

R_f is typically a potentiometer, which can be adjusted to bring the circuit to the oscillation condition, Adjusting so that $R_f = 2R_1$ brings the circuit to the onset of oscillation. Making R_f *smaller* guarantees oscillation, at the cost of linearity of the sinusoid.

Because v_{o1} is the integral of v_{o2}, it will be shifted in phase by 90°. Also, the filtering action of the integrator circuit tends to make v_{o1} more linear (i.e. have less distortion) than v_{o2}.