second-order active filters

Building up higher-order active filters from first-order filters is OK, but limiting, because we can never have $Q_P > 0.5$ by using first-order building blocks. To get more flexibility we need slightly different approaches. Here we describe three types of second-order active filter circuits that can achieve higher Q_P.

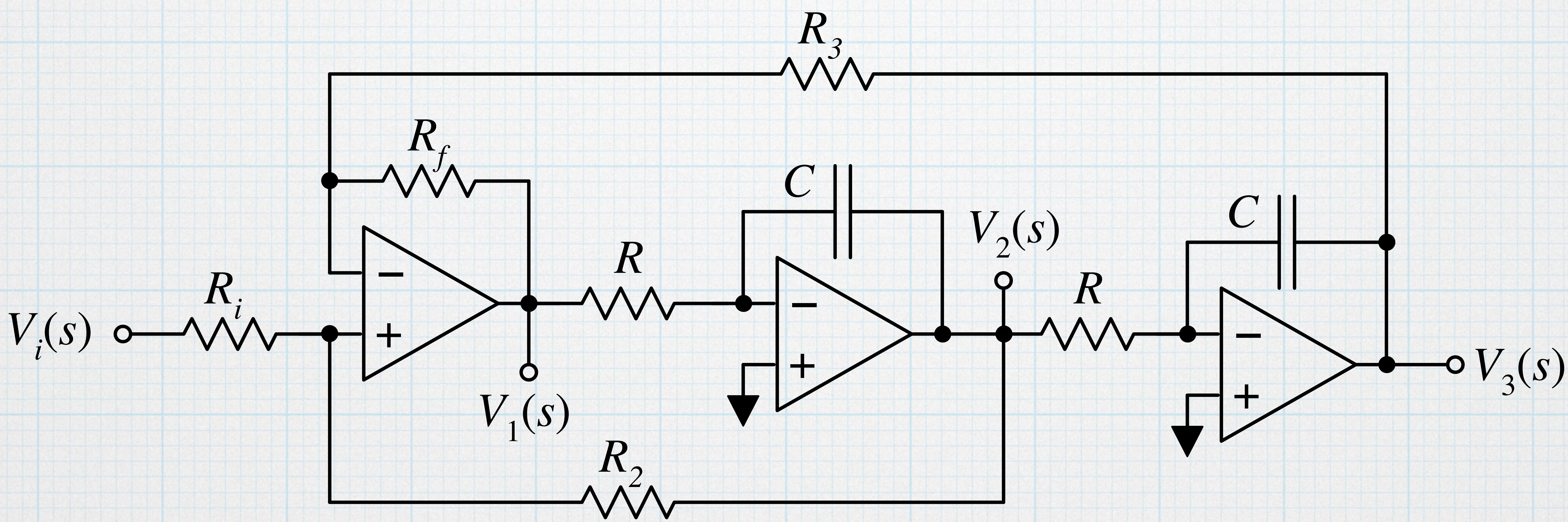
Three examples

1. Two integrator-loop biquad (Kerwin-Huelsman-Newcomb). Uses three op-amps. Good for any type of filter. Use a fourth op-amp to adjust gain.
2. Delyiannis-Friend single amp biquad (SAB). Uses one op-amp. Works well for band-pass. Any value of Q_P.
3. Sallen-Key single amp biquad. Again, uses one op-amp. Works well for low-pass and high-pass. Any value of Q_P.

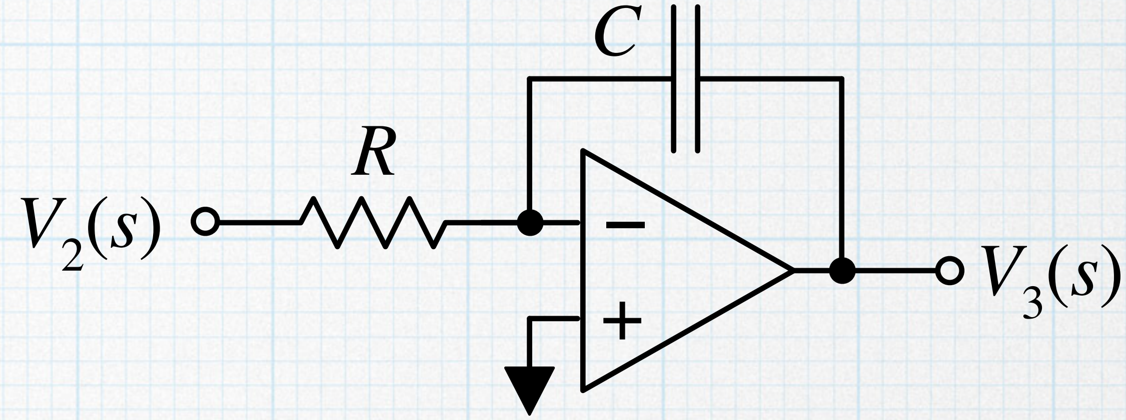
Two-integrator loop

Kerwin-Huelsman-Newcomb (KHN) biquad

Here is a crazy looking circuit. Let's analyze it to find the transfer function. As usual, tackle the problem piece by piece.

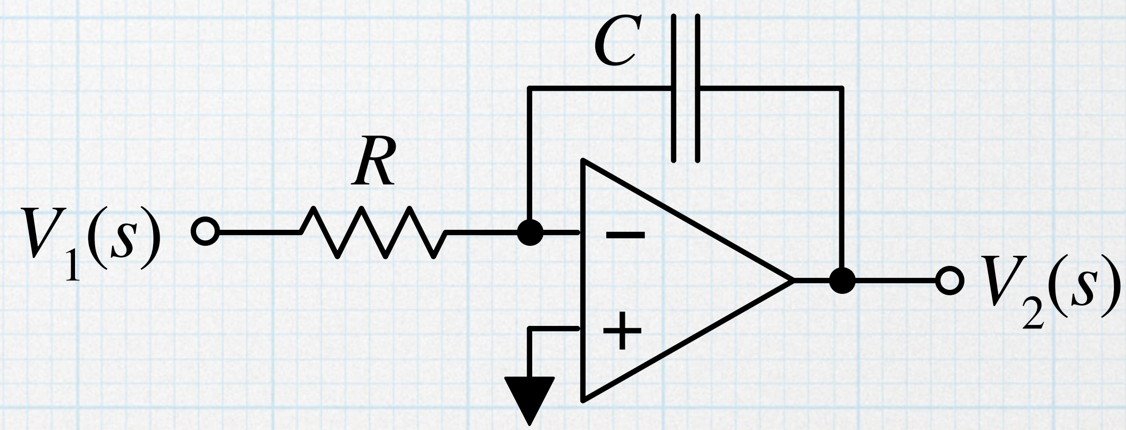


1. Start on the right. We note that the third op-amp is a simple integrator. In the Laplace domain:



$$V_3(s) = -\frac{V_2(s)}{sRC}$$

2. Same thing for the second op amp.

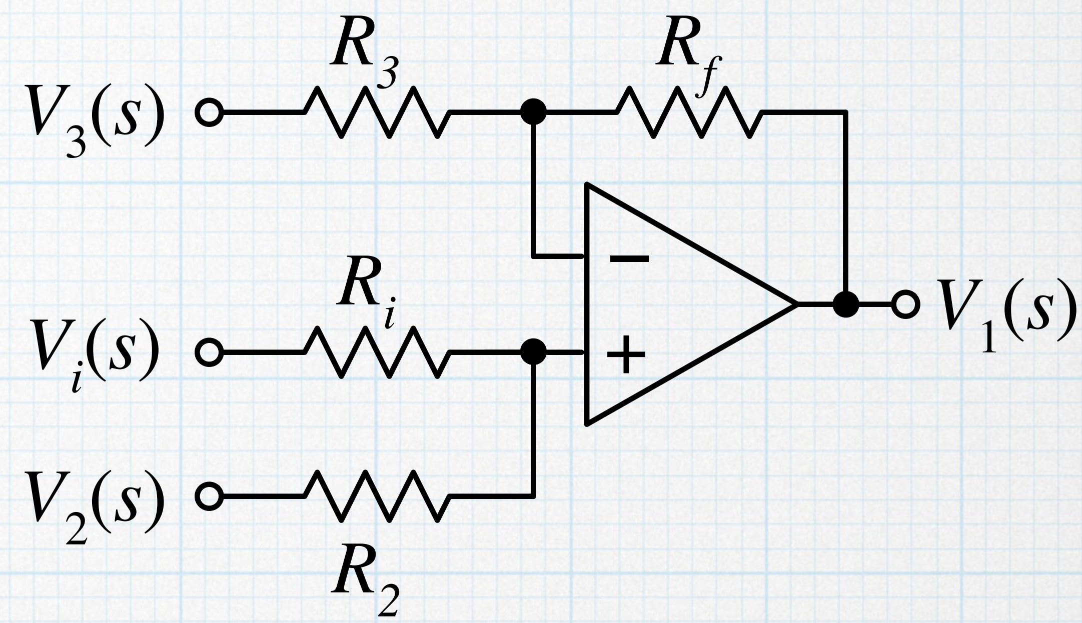


$$V_2(s) = -\frac{V_1(s)}{sRC}$$

So V_3 is essential the second integral of V_1.

$$V_3(s) = \frac{V_1(s)}{(sRC)^2}$$

3. The first op amp is some sort of summing circuit. V_2 and V_3 are being fed back and combined with V_i in some fashion. We have seen circuits like this before. (It could have been a homework problem. It is straight-forward — and tedious — to show that:



$$V_1(s) = \left(\frac{1 + \frac{R_f}{R_3}}{1 + \frac{R_i}{R_2}} \right) V_i(s) + \left(\frac{1 + \frac{R_f}{R_3}}{1 + \frac{R_2}{R_i}} \right) V_2(s) - \left(\frac{R_f}{R_3} \right) V_3(s)$$

$$= G_i V_i(s) + G_2 V_2(s) - G_3 V_3(s)$$

$$V_1(s) = G_i V_i(s) - \frac{G_2}{sRC} V_1(s) - \frac{G_3}{(sRC)^2} V_1(s)$$

$$V_1(s) = G_i V_i(s) - \frac{G_2}{sRC} V_1(s) - \frac{G_3}{(sRC)^2} V_1(s)$$

With a bit of re-arrangement, we can re-cast this in the form of a transfer function relating $V_1(s)$ to $V_i(s)$.

$$T_1(s) = \frac{V_1(s)}{V_i(s)} = G_i \cdot \frac{s^2}{s^2 + \left(\frac{G_2}{RC}\right) s + \frac{G_3}{(RC)^2}}$$

Of course, this is a high-pass filter with $\omega_o^2 = G_3 / (RC)^2$, $Q_P = (\omega_o RC) / G_2$, and $G_o = G_i$.

By choosing the RC product in the integrators and the resistor ratios that determine the various gains, we can design for ω_o and Q_P. With all of the components, it would seem that there should be enough freedom that we could specify the gain, as well. However, the resistor ratios are a somewhat constrained, and we aren't able to choose all three parameters independently. Basically, you can choose two out of three. If a specific gain is needed, a non-inverting amp can be added at the output to provide extra gain.

But there's more. Recall that:

$$V_2(s) = -\frac{V_1(s)}{sRC}$$
$$= -G_i \cdot \frac{\left(\frac{1}{RC}\right)s}{s^2 + \left(\frac{G_2}{RC}\right)s + \frac{G_3}{(RC)^2}} V_i(s)$$

Expressing this in the from of a transfer function:

$$T_2(s) = \frac{V_2(s)}{V_i(s)} = -\frac{G_i}{G_2} \cdot \frac{\left(\frac{G_2}{RC}\right)s}{s^2 + \left(\frac{G_2}{RC}\right)s + \frac{G_3}{(RC)^2}}$$

We see that this is a high-pass filter with $\omega_o^2 = G_3/(RC)^2$ and $Q_P = (\omega_o RC)/G_2$, and $G_o = -G_i/G_2$.

There's even more where that came from.

$$V_3(s) = -\frac{V_2(s)}{sRC}$$

$$= G_i \cdot \frac{\frac{1}{(RC)^2}}{s^2 + \left(\frac{G_2}{RC}\right)s + \frac{G_3}{(RC)^2}} V_i(s)$$

Expressing this in the from of a transfer function:

$$T_3(s) = \frac{V_3(s)}{V_i(s)} = \frac{G_i}{G_3} \cdot \frac{\frac{G_3}{(RC)^2}}{s^2 + \left(\frac{G_2}{RC}\right)s + \frac{G_3}{(RC)^2}}$$

The third output is a low-pass filter with $\omega_o^2 = G_3/(RC)^2$ and $Q_P = (\omega_o RC)/G_2$, and $G_o = G_i/G_3$.

It's pretty nifty — one circuit that can be used for all three primary types of second-order filters.

KHN summary

For all of the sections:

$$\omega_o^2 = \frac{G_3}{(RC)^2} \qquad Q_P = \frac{\omega_o RC}{G_2} = \frac{G_3}{G_2}$$

$$G_i = \left(\frac{1 + \frac{R_f}{R_3}}{1 + \frac{R_i}{R_2}}\right) \qquad G_2 = \left(\frac{1 + \frac{R_f}{R_3}}{1 + \frac{R_2}{R_i}}\right) \qquad G_3 = \frac{R_f}{R_3}$$

For LP: $G_o = G_i / G_3$

For BP: $G_o = -G_i / G_2$

For HP: $G_o = G_i$

Note that G_i and G_2 are not independent, $G_2 = (R_i / R_2) \cdot G_i$. (Check it.) This means that for LP and HP, there is not enough flexibility in the components to choose ω_o, Q_P, and G_o completely independently. So typically, the basic KHN would be designed for ω_o and Q_P, and some value for G_o will come from that. If a specific gain is required, a fourth amp — inverting or non-inverting — can be added to adjust G_o.

KHN design example – LP

Design a KHN low-pass filter with corner frequency of 10 kHz and gain in the passband of 5. The realized circuit should meet the specs within ±5%.

1. Since no Q_P was specified, presumably we can choose what we want. To keep it simple, choose $Q_P = 0.707$, which is the maximally flat case and for which $\omega_c = \omega_o$.

2. Design for $\omega_o = 2\pi f_o = 62.83$ krad/s.

3. $\omega_o = \frac{\sqrt{G_3}}{RC}$. To keep it simple, choose $G_3 = 1$ ($R_3 = R_f$.) With a bit of calculator poking, we note that the combination of a 3.3-kΩ resistor and a 4.7-nF capacitor gives $\omega_o = (RC)^{-1} = 122.5$ krad/s. (–2.5%).

4. Since we chose $G_3 = 1$, then $Q_P = \frac{1}{G_2} \rightarrow G_2 = \frac{1}{Q_P} = 1.41$

KHN low-pass design (con't)

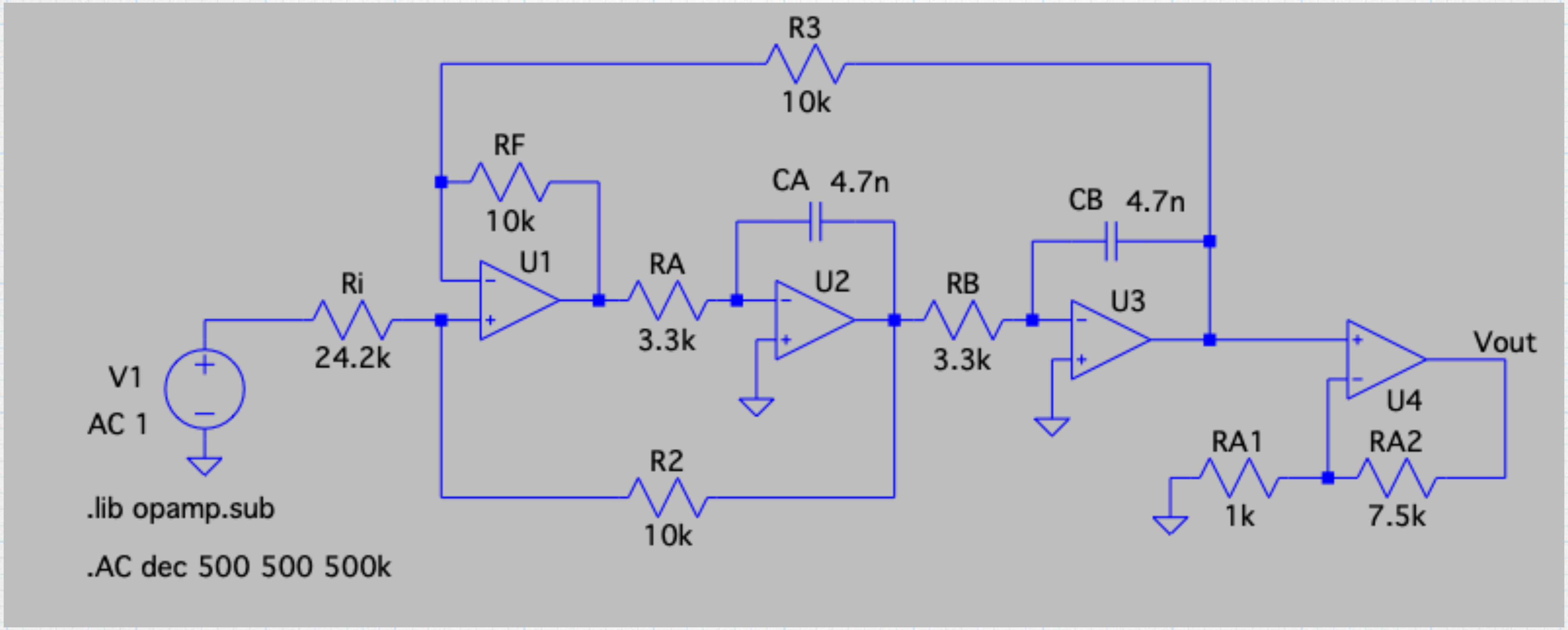
5. We have already chosen $R_3 = R_f$, so then the G_2 expression is

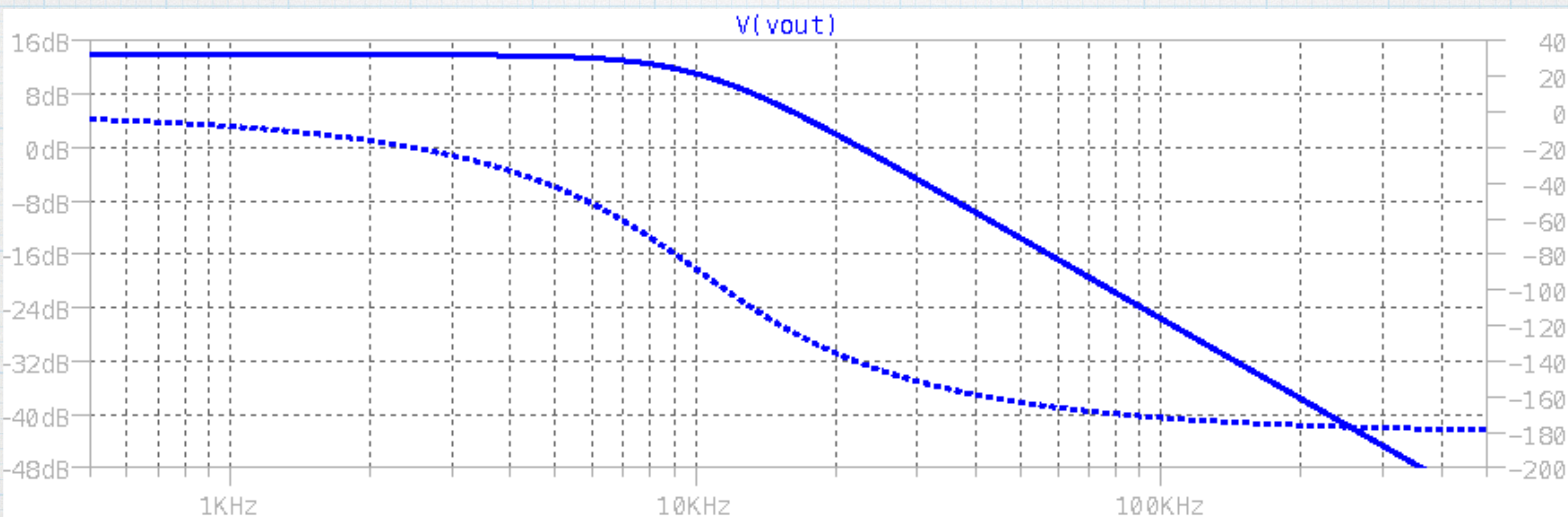
$$G_2 = \left(\frac{2}{1 + \frac{R_2}{R_i}} \right) \quad \rightarrow \quad \frac{R_2}{R_i} = \frac{2}{G_2} - 1 = 0.418$$

   This is a bit harder to find simple resistor ratios that are close. But a bit of trial-and-error shows that if R_2 = 10 kΩ and R_i = 24.2 kΩ, the ratio is 0.413, which is about 1% off. R_i can be made with 22 kΩ in series with a 2.2 kΩ.

6. Finally, with $G_3 = 1$, $G_o = G_i = \left(\frac{2}{1 + \frac{R_i}{R_2}} \right) = 0.59$. Since the requirement is for a gain of 5 in the passband, an extra amp with gain of 5/0.59 = 8.5 must be added after the LP output.

Because of the relatively high frequencies in the LTspice simulation on the following page, the gain-bandwidth of the op-amps was increased 1 GHz so that GBW limitations did not affect the Bode plots.


R3
10k
RF
10k
CA 4.7n
CB 4.7n
U1
RA
3.3k
U2
RB
3.3k
U3
Vout
Ri
24.2k
V1
AC 1
U4
RA1
1k
RA2
7.5k
R2
10k
.lib opamp.sub
.AC dec 500 500 500k


V(vout)
16dB
8dB
0dB
-8dB
-16dB
-24dB
-32dB
-40dB
-48dB
40°
20°
0°
-20°
-40°
-60°
-80°
-100°
-120°
-140°
-160°
-180°
-200°
1KHz
10KHz
100KHz

KHN design example – HP

Design a KHN high-pass filter with corner frequency of 250 Hz and gain in the passband of 1. Design for $Q_p = 0.6$. The realized circuit should meet the specs within ±5%.

1. Since $Q_P \neq 0.707$, then $\omega_c \neq \omega_o$, and we must use the messy equation to find the required value of ω_o. In the high-pass case

$$\omega_o = \omega_c \sqrt{1 - \frac{1}{2Q_P^2} + \sqrt{1 + \left(1 - \frac{1}{2Q_P^2}\right)^2}}$$

   With $Q_P = 0.6$ and $\omega_o = 0.827 \cdot \omega_c = 1300$ rad/s — this is the characteristic frequency that we will design for.

2. $\omega_o = G_3 / RC$. To keep it simple, choose $G_3 = 1$ ($R_3 = R_f$.) With a bit of calculator poking, we note that the combination of a 7.5-kΩ resistor and a 0.1-μF capacitor gives $\omega_o = (RC)^{-1} =$ 1333 rad/s — +2.5% off. (7.5-kΩ resistors are available. Or it can realized with a 6.8 kΩ and 680 kΩ in series.)

KHN high-pass design (con't)

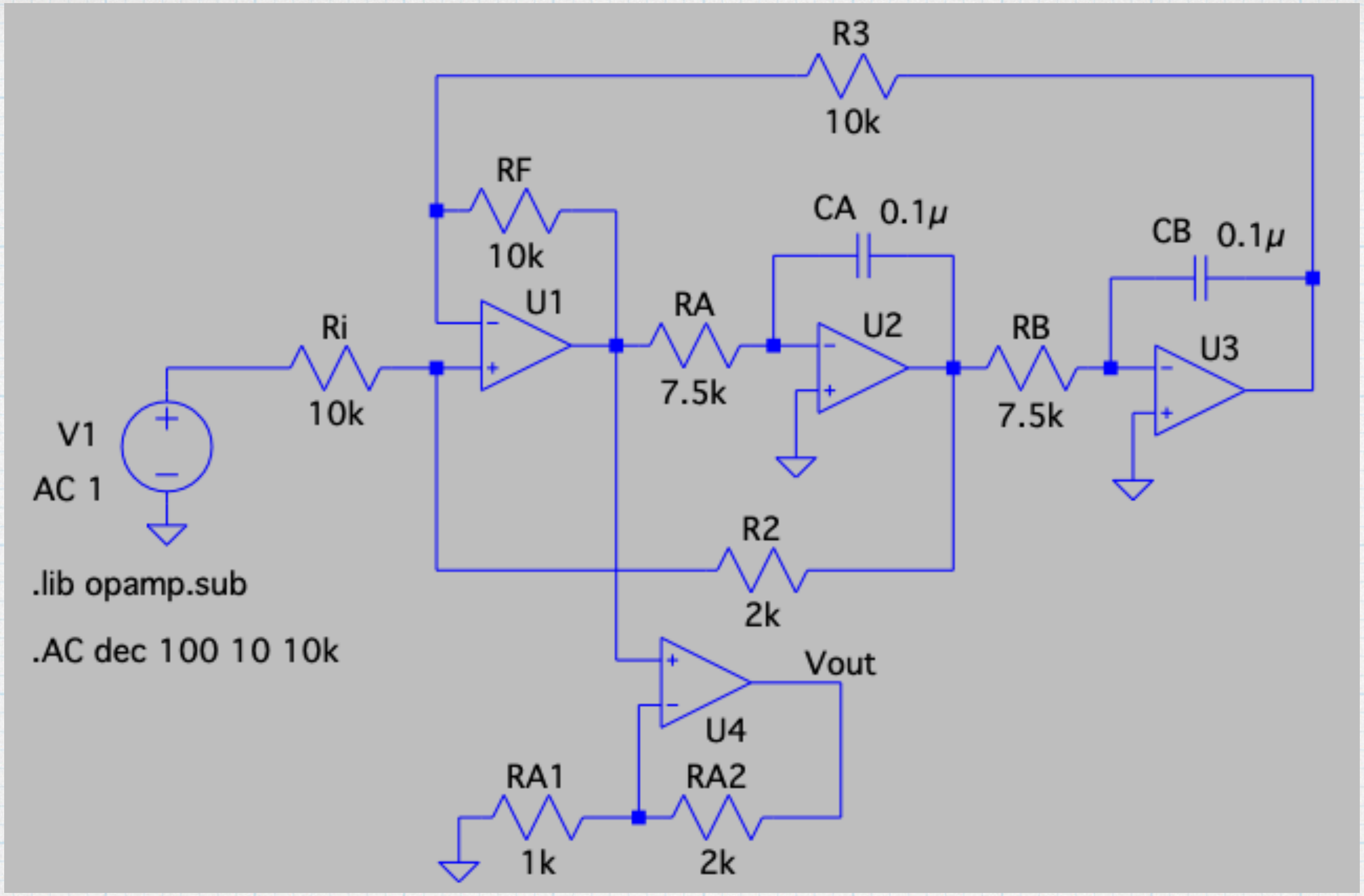
3. Since we chose $G_3 = 1$, then $Q_P = \frac{1}{G_2}$ $\rightarrow$ $G_2 = \frac{1}{Q_P} = 1.67$

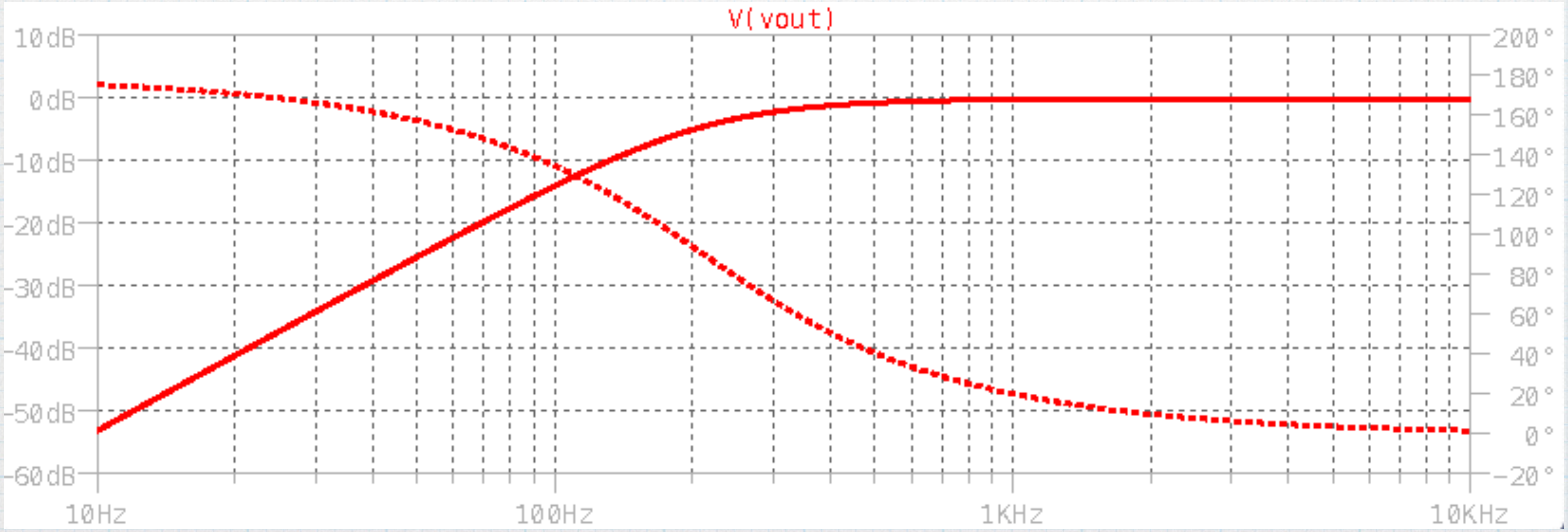
4. We have already chosen $R_3 = R_f$, so then the G_2 expression is

$$G_2 = \left(\frac{2}{1 + \frac{R_2}{R_i}} \right) \rightarrow \frac{R_2}{R_i} = \frac{2}{G_2} - 1 = 0.2$$

   Choose $R_2 = 10\ \mathrm{k}\Omega$ and $R_i = 2\ \mathrm{k}\Omega$ (two 1-kΩ resistors in series).

5. Finally, for high-pass, $G_o = G_i = \left(\frac{1 + \frac{R_f}{R_3}}{1 + \frac{R_i}{R_2}} \right)$. These ratios are already specified, giving $G_o = 0.333$. So to meet the pass-band gain requirement, we need an extra amp with gain of 3 — easily accomplished.


R3
10k
RF
10k
CA
0.1μ
CB
0.1μ
U1
RA
7.5k
U2
RB
7.5k
U3
Ri
10k
V1
AC 1
R2
2k
.lib opamp.sub
.AC dec 100 10 10k
Vout
U4
RA1
1k
RA2
2k


V(vout)
10dB
0dB
-10dB
-20dB
-30dB
-40dB
-50dB
-60dB
200°
180°
160°
140°
120°
100°
80°
60°
40°
20°
0°
-20°
10Hz
100Hz
1KHz
10KHz

KHN design example – BP

Design a KHN band-pass filter with center frequency of 2400 Hz and bandwidth of 400 Hz. The gain of the signal at the center frequency should be 10. The realized circuit should meet the specs within ±5%.

Note, because of the particular form for G_o for the KHN BP, it is possible to meet all design parameters with just the three amps — no extra amp is needed in order to adjust gain at the end. An extra amp can be added if desired, but is it not a requirement in order to set the correct gain as in the case of the LP and HP.

1. $\omega_o = 2\pi f_o = 15.08$ krad/s.

2. $\omega_o / Q_P = \Delta\omega$ (bandwidth) → $Q_P = f_o / \Delta f = 2400/400 = 6$.

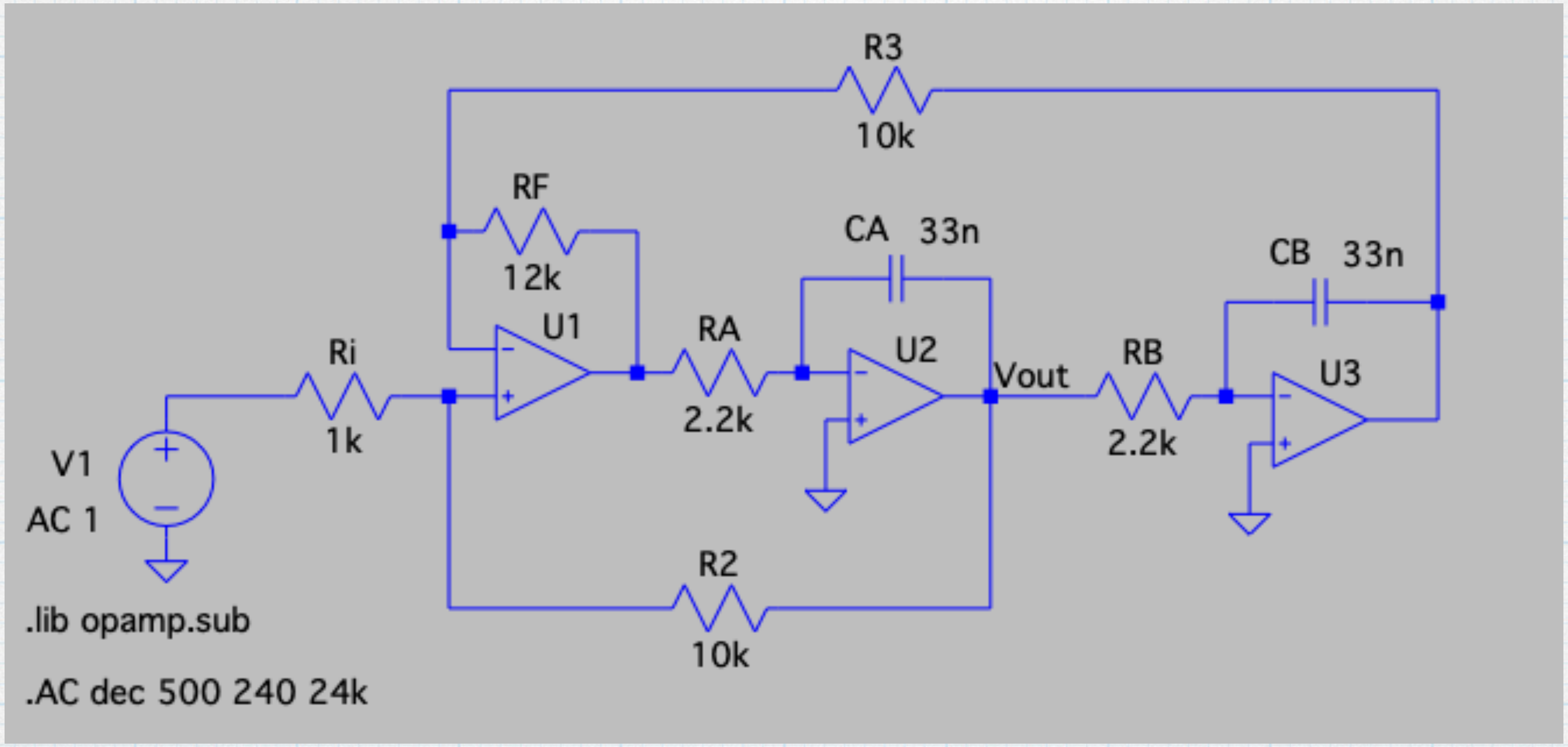
3. For the BP, $G_o = -G_i/G_2 = -R_2/R_i$. So to meet the gain requirement, set $R_2 = 10R_i$. (Choose 10 kΩ and 1 kΩ.)

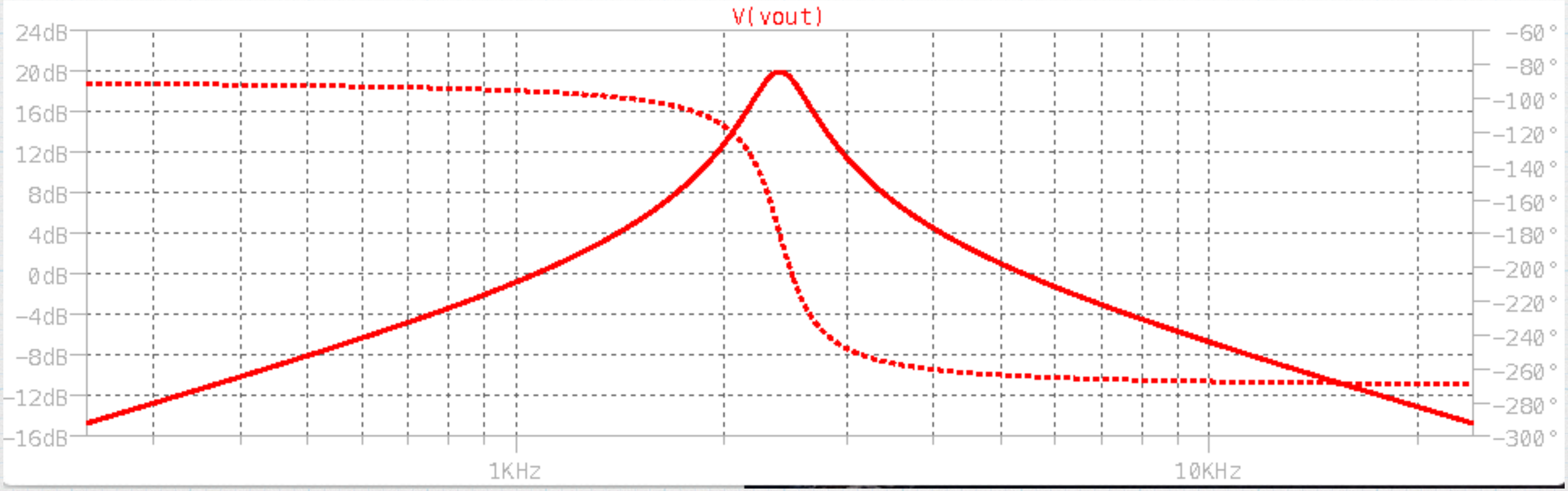
KHN band-pass design (con't)

4. $$Q_P = \frac{G_3}{G_2} = \frac{\left(R_f/R_3\right)}{\left(\dfrac{1+R_f/R_3}{1+R_2/R_i}\right)} = \frac{1+R_2/R_i}{1+R_3/R_f}$$

5. Since Q_P must be 6 and we've already chosen $R_2/R_i = 10$, we can solve the above equation for $R_3/R_f = 0.833$. ($R_f/R_3 = 1.2$.) A combination of $R_f = 12$ kΩ (a fairly common value) and $R_3 = 10$ kΩ works nicely.

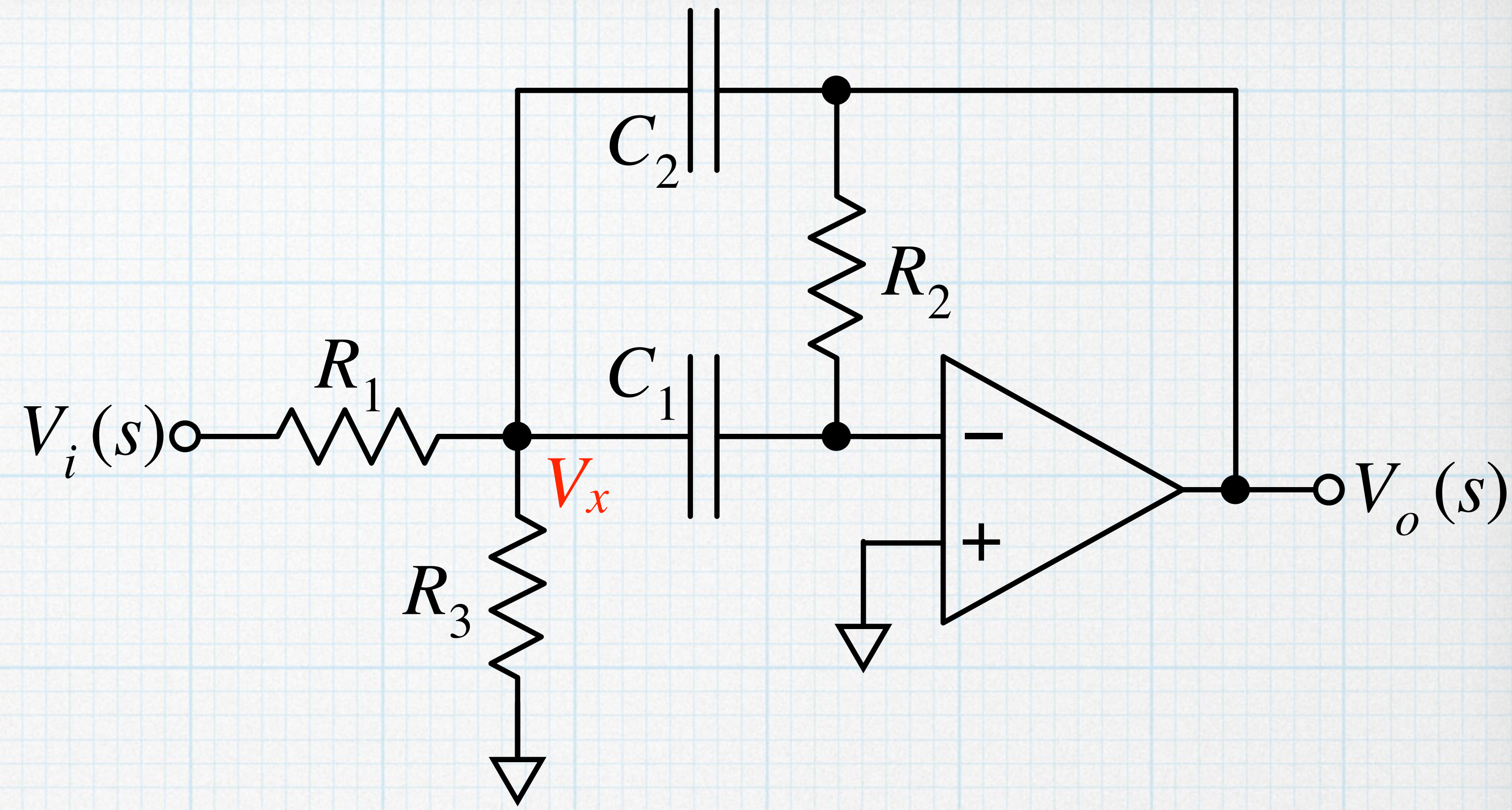
6. Finally, $\omega_o = \sqrt{G_3}/(RC) \rightarrow RC = \sqrt{G_3}/\omega_o = \sqrt{1.2}/\left(15.08\,\mathrm{krad/s}\right)$. A combination of $R = 2.2$ kΩ and $C = 33$ nF gives $\omega_o = 15.09$ krad/s — essentially on the mark.


R3
10k
RF
12k
CA 33n
CB 33n
U1
RA
2.2k
U2
Vout
RB
2.2k
U3
Ri
1k
V1
AC 1
R2
10k
.lib opamp.sub
.AC dec 500 240 24k


V(vout)
24dB
20dB
16dB
12dB
8dB
4dB
0dB
-4dB
-8dB
-12dB
-16dB
-60°
-80°
-100°
-120°
-140°
-160°
-180°
-200°
-220°
-240°
-260°
-280°
-300°
1KHz
10KHz

Delyiannis-Friend

The Delyiannis-Friend configuration makes a nice single amp bandpass circuit. The three resistors allow for flexibility in choosing ω_o, Q_p, and G_o.



Calculate the transfer function. Write node-voltage equations at the inverting input (virtual ground) and the node labeled V_x.

$$sC_1V_x = \frac{-V_o}{R_2} \qquad \frac{V_i - V_x}{R_1} = \frac{V_x}{R_3} + sC_1V_x + sC_2\left(V_x - V_o\right)$$

$$V_x = -\frac{V_o}{sR_2C_1} \qquad V_i + sR_1C_2V_o = \left(1 + \frac{R_1}{R_3} + sR_1C_1 + sR_1C_2\right)V_x$$

Substitute to eliminate V_x.

$$V_i + sR_1C_2V_o = -\left(1 + \frac{R_1}{R_3} + sR_1C_1 + sR_1C_2\right)\frac{V_o}{sR_2C_1}$$

Grind through the algebra to arrive at the standard band-pass form.

$$V_i = -\left(sR_1C_2 + \frac{1}{sR_2C_1} + \frac{R_1}{sR_2R_3C_1} + \frac{R_1}{R_2} + \frac{R_1C_2}{R_2C_1} \right) V_o$$

$$T(s) = \frac{V_o}{V_i} = -\frac{1}{\left(sR_1C_2 + \frac{1}{sR_2C_1} + \frac{R_1}{sR_2R_3C_1} + \frac{R_1}{R_2} + \frac{R_1C_2}{R_2C_1} \right)}$$

$$= -\frac{\left(\frac{1}{R_1C_2} \right) s}{\left(s^2 + \frac{1}{R_1R_2C_1C_2} + \frac{1}{R_2R_3C_1C_2} + \frac{s}{R_2C_2} + \frac{s}{R_2C_1} \right)}$$

$$T(s) = -\frac{\left(\frac{1}{R_1C_2} \right) s}{s^2 + \left(\frac{1}{R_2C_2} + \frac{1}{R_2C_1} \right) s + \left(\frac{1 + \frac{R_1}{R_3}}{R_1R_2C_1C_2} \right)}$$

$$\omega_o^2 = \frac{1 + \frac{R_1}{R_3}}{R_1R_2C_1C_2} \qquad \frac{\omega_o}{Q_P} = \frac{1}{R_2C_1} + \frac{1}{R_2C_2} \qquad G_o\frac{\omega_o}{Q_p} = \frac{1}{R_1C_2}$$

The three transfer-function parameters can be expressed directly in terms of the circuit components.

$$\omega_o^2 = \frac{1+\frac{R_1}{R_3}}{R_1 R_2 C_1 C_2} \qquad Q_P = \frac{\omega_o}{\frac{1}{R_2 C_1}+\frac{1}{R_2 C_2}} = \frac{\omega_o R_2 C_2}{1+\frac{C_2}{C_1}} = \frac{\sqrt{\frac{R_2}{R_1}+\frac{R_2}{R_3}}}{\sqrt{\frac{C_1}{C_2}}+\sqrt{\frac{C_2}{C_1}}}$$

$$G_o = \frac{Q_P}{\omega_o R_1 C_2} = \frac{\frac{R_2}{R_1}}{1+\frac{C_2}{C_1}}$$

These equations are fine for *analyzing* a D-F circuit but are unwieldy for *designing* a circuit. Trying to adjust five circuit parameters to obtain specific values of ω_o, Q_p, and G_o will be inefficient, at best. To make the design process easier, we can impose a constraint up front. For the D-F filter, a commonly used constraint is to make the two capacitors equal, $C_1 = C_2 = C$. The above equations are simplified, and we can choose the resistors in a straight-forward manner.

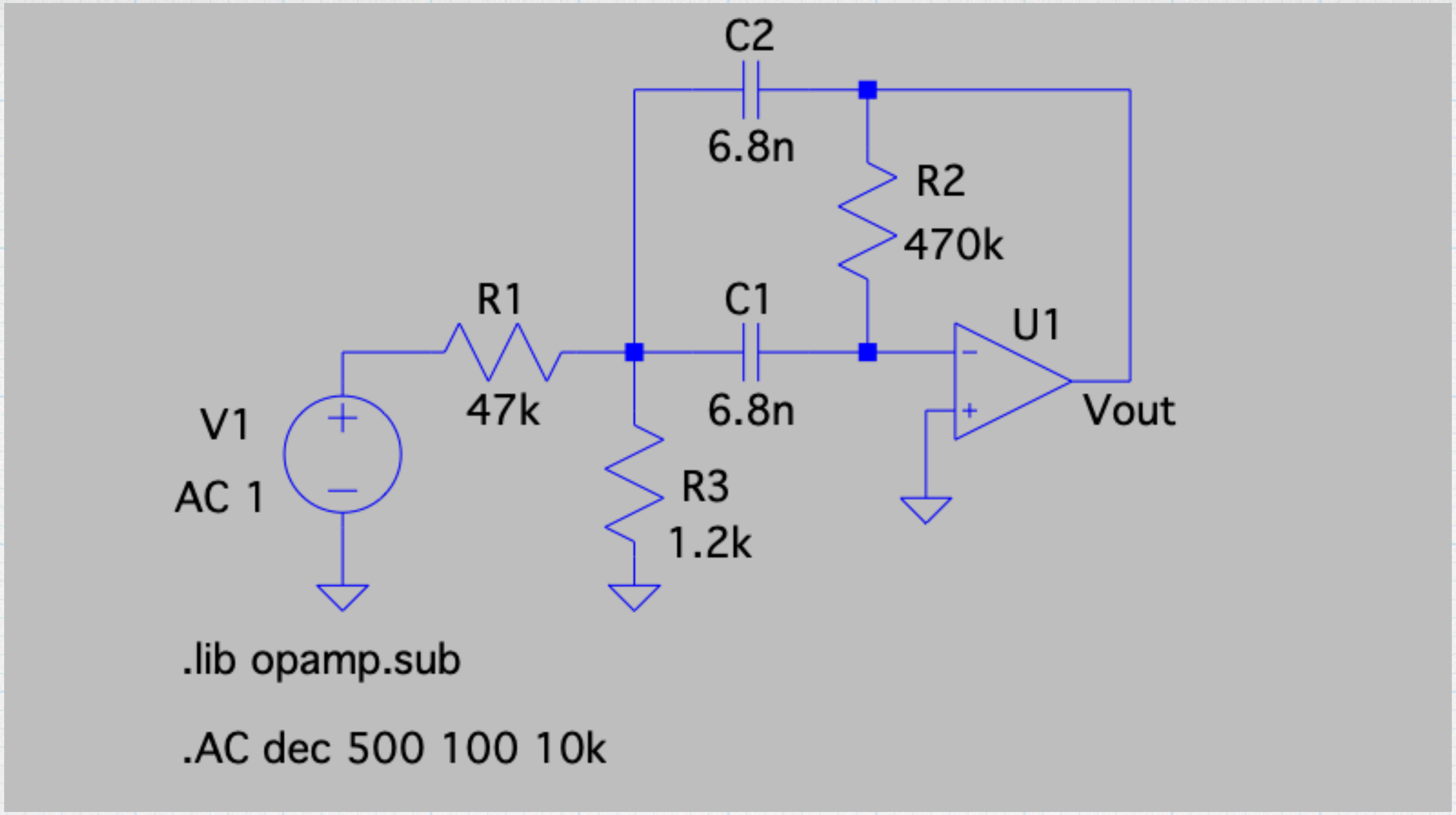
$$\omega_o^2 = \frac{1+\frac{R_1}{R_3}}{R_1 R_2 C^2} \qquad Q_P = \frac{1}{2}\omega_o R_2 C \qquad G_o = \frac{R_2}{2R_1}$$

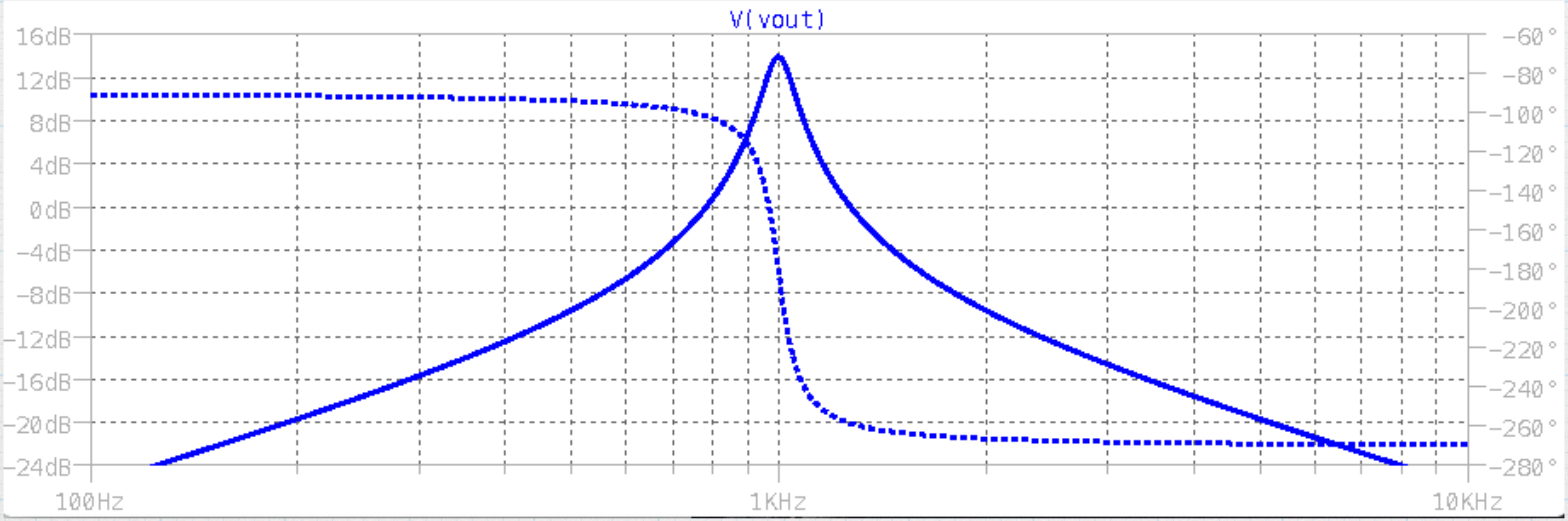
A design example follows.

Example

Design a D-F bandpass filter with center frequency of 1000 Hz, bandwidth of 100 Hz, and band-center gain of 5. The design should meet the specifications within 5%.

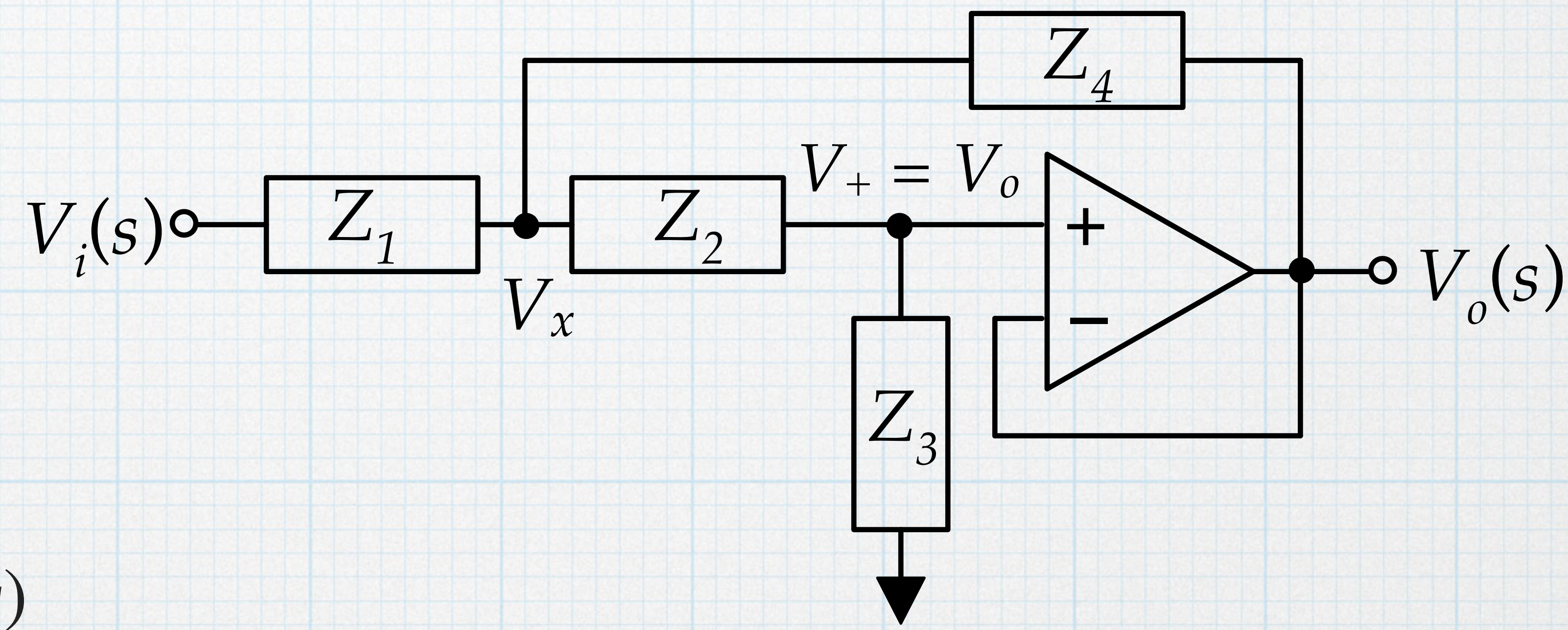
1. $\omega_o = 2\pi f_o = 6280$ rad/s.
2. $\omega_o / Q_P = \Delta\omega$ (bandwidth) $\rightarrow$ $Q_P = f_o / \Delta f = 10$.
3. Choose $C_1 = C_2 = 6.8$ nF.
4. $Q_P = \dfrac{1}{2}\omega_o R_2 C \quad \rightarrow \quad R_2 = \dfrac{2Q_P}{\omega_o C} = 468\,\mathrm{k\Omega}$
5. $G_o = \dfrac{R_2}{2R_1} \quad \rightarrow \quad R_1 = \dfrac{R_2}{2G_o} = 46.8\,\mathrm{k\Omega}$
6. $\omega_o^2 = \dfrac{1 + \frac{R_1}{R_3}}{R_1 R_2 C^2} \quad \rightarrow \quad R_3 = \dfrac{R_1}{\omega_o^2 R_1 R_2 C^2 - 1} = 1.20\,\mathrm{k\Omega}$
7. Choosing standard values, we can build the D-F circuit with $R_1 = 47$ kΩ, $R_2 = 470$ kΩ, $R_1 = 1.2$ kΩ, and $C_1 = C_2 = 6.8$ nF.


C2
6.8n
R2
470k
R1
47k
C1
6.8n
U1
V1
AC 1
Vout
R3
1.2k
.lib opamp.sub
.AC dec 500 100 10k


V(vout)
16dB
12dB
8dB
4dB
0dB
-4dB
-8dB
-12dB
-16dB
-20dB
-24dB
-60°
-80°
-100°
-120°
-140°
-160°
-180°
-200°
-220°
-240°
-260°
-280°
100Hz
1KHz
10KHz

Sallen-Key

Another single-amp bi-quad is the Sallen-Key circuit. Although it looks similar to a D-F filter, it is fundamentally different. The amp is configured in a non-inverting fashion — usually with unity gain. The general form of the circuit is shown below.



To find the transfer function, we start by noting that the unity-gain feedback means that

$$V_+(s) = V_-(s) = V_o(s)$$

Writing node equations:

$$\frac{V_i - V_x}{Z_1} = \frac{V_x - V_o}{Z_2} + \frac{V_x - V_o}{Z_4}$$

(Effectively, Z_2 and Z_4 are in parallel.)

$$\frac{V_x - V_o}{Z_2} = \frac{V_o}{Z_3}$$

Re-arranging the equations:

$$V_i = \left(1 + \frac{Z_1}{Z_2} + \frac{Z_1}{Z_4}\right) V_x - \left(\frac{Z_1}{Z_2} + \frac{Z_1}{Z_4}\right) V_o$$

$$V_x = \left(1 + \frac{Z_2}{Z_3}\right) V_o$$

Substitute the second into the first to eliminate V_x

$$V_i = \left(1 + \frac{Z_1}{Z_2} + \frac{Z_1}{Z_4}\right)\left(1 + \frac{Z_2}{Z_3}\right) V_o - \left(\frac{Z_1}{Z_2} + \frac{Z_1}{Z_4}\right) V_o$$

Multiply everything out

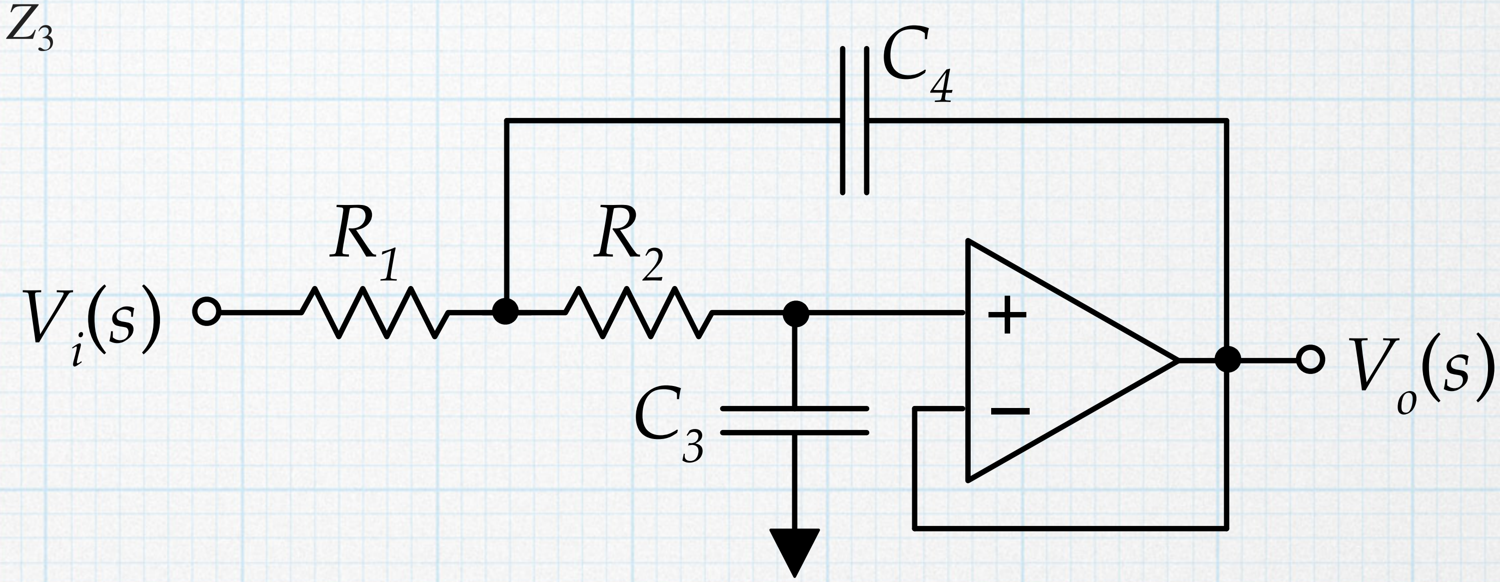
$$V_i = \left(1 + \frac{Z_1}{Z_2} + \frac{Z_1}{Z_4} + \frac{Z_1}{Z_3} + \frac{Z_2}{Z_3} + \frac{Z_1}{Z_4}\frac{Z_2}{Z_3} - \frac{Z_1}{Z_2} - \frac{Z_1}{Z_4}\right) V_o$$

And wrangle it into a transfer function

$$T(s) = \frac{V_o}{V_i} = \frac{1}{1 + \frac{Z_1}{Z_3} + \frac{Z_2}{Z_3} + \frac{Z_1}{Z_4}\frac{Z_2}{Z_3}}$$

$$T(s) = \frac{1}{1 + \frac{Z_1}{Z_3} + \frac{Z_2}{Z_3} + \frac{Z_1}{Z_4}\frac{Z_2}{Z_3}}$$

It doesn't mean much until specific impedances are inserted. Consider the circuit shown at right.



$$Z_1 = R_1 \quad Z_2 = R_2 \quad Z_3 = \frac{1}{sC_3} \quad Z_4 = \frac{1}{sC_4}$$

$$T(s) = \frac{1}{1 + sR_1C_3 + sR_2C_3 + s^2R_1R_2C_3C_4}$$

$$= \frac{\frac{1}{R_1R_2C_3C_4}}{s^2 + s\left(\frac{1}{R_1C_4} + \frac{1}{R_2C_4}\right) + \frac{1}{R_1R_2C_3C_4}}$$

Make it high-pass by swapping resistors and capacitors.

Clearly low-pass with: $\omega_o^2 = \frac{1}{R_1R_2C_3C_4}$ $\quad \frac{\omega_o}{Q_P} = \frac{1}{R_1C_4} + \frac{1}{R_2C_4}$

Sallen-Key: design example

Design a Sallen-Key second-order low-pass filter that is maximally flat with corner frequency at 1000 Hz.

Maximally flat means that the filter should have $Q_P = 0.707 \left(1/\sqrt{2}\right)$. Recall that this condition corresponds to the special case in which $\omega_c = \omega_o$. So we need a low-pass filter with $f_o = 1000$ Hz ($\omega_o = 6283$ rad/s).

Using the Sallen-Key equations.

$$\omega_o^2 = \frac{1}{R_1 R_2 C_3 C_4} \qquad \omega_o = \frac{1}{\sqrt{R_1 R_2 C_3 C_4}}$$

$$\frac{\omega_o}{Q_P} = \frac{1}{R_1 C_4} + \frac{1}{R_2 C_4} \qquad Q_P = \frac{\frac{1}{\sqrt{R_1 R_2 C_3 C_4}}}{\frac{1}{R_1 C_4} + \frac{1}{R_2 C_4}}$$

Those are a little messy. Let's impose a constraint in an effort to simplify things: Let's choose to have $R_1 = R_2 = R$.

Then the equations for ω_o and Q_P become:

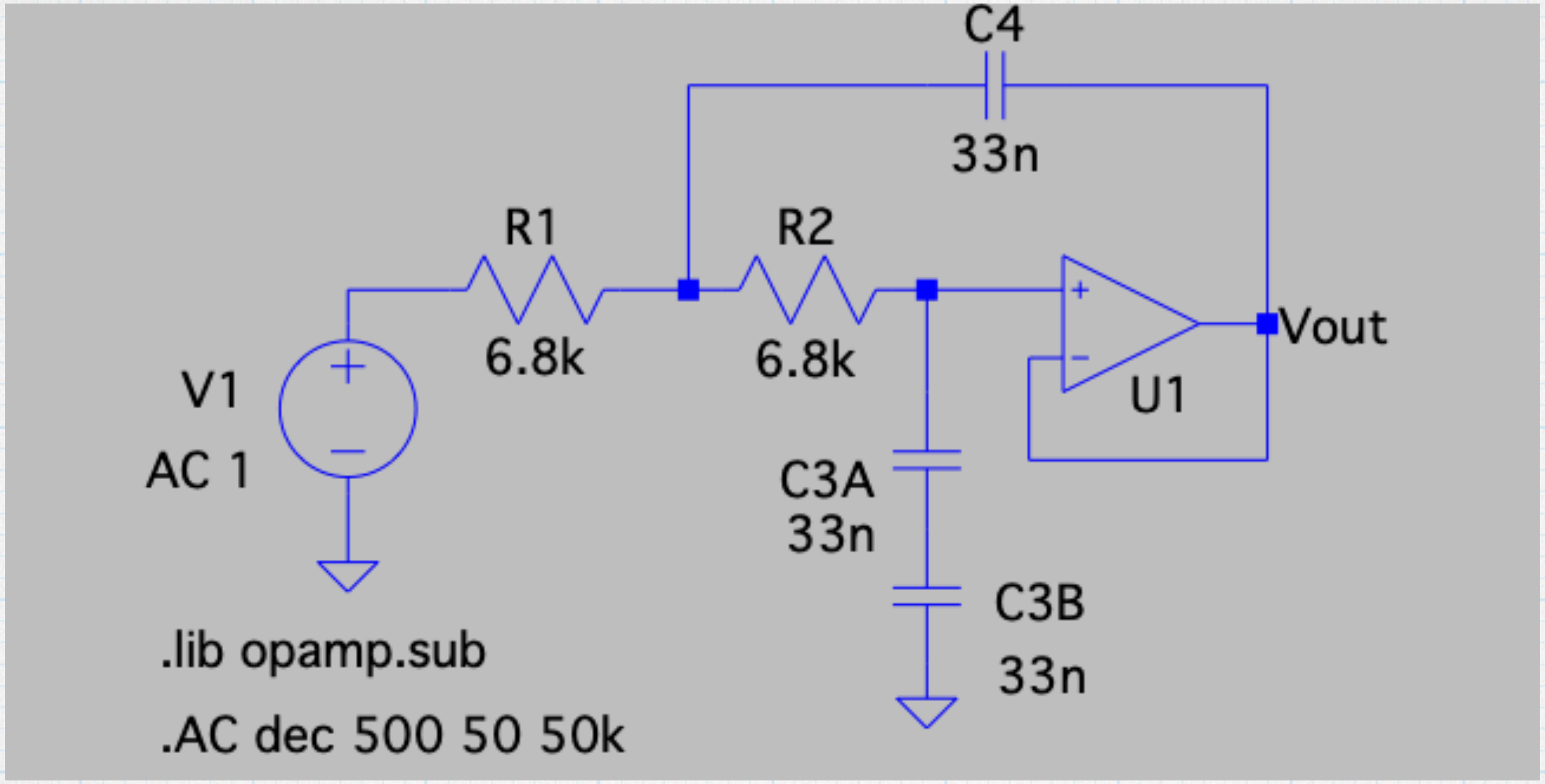
$$\omega_o = \frac{1}{R\sqrt{C_3 C_4}} \qquad Q_P = \frac{1}{2}\sqrt{\frac{C_4}{C_3}}$$

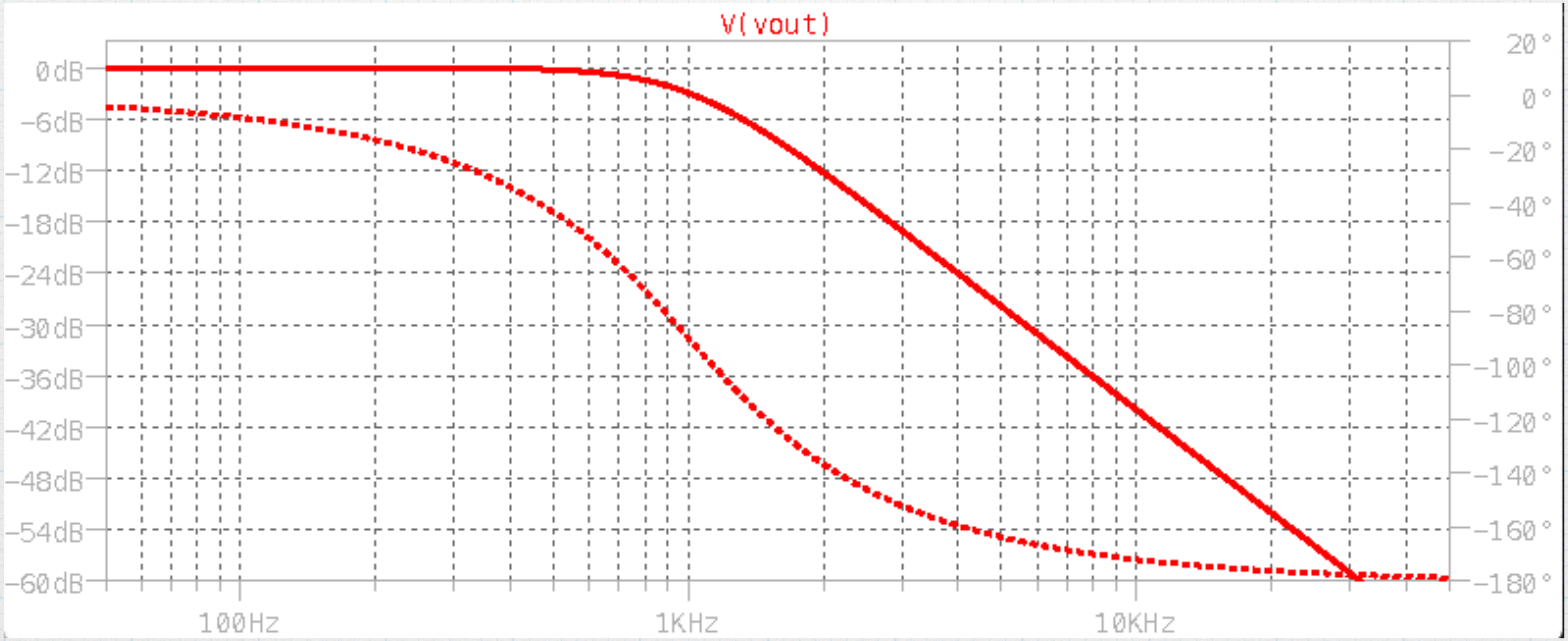
We can meet the quality factor requirement by choosing $C_4 = 2C_3$.

$$Q_P = \frac{\sqrt{2}}{2} = 0.707$$

Then: $\omega_o = \dfrac{1}{\sqrt{2}RC_3}$.

With a bit of noodling around around with a calculator, we see that the combination of R = 6.8 kΩ and C_3 = 16.5 nF works. Then C_4 = 33 nF (a standard size) and C_3 could be made with two 33 nF caps in series.


C4
33n
R1
6.8k
R2
6.8k
V1
AC 1
Vout
U1
C3A
33n
C3B
33n
.lib opamp.sub
.AC dec 500 50 50k


V(vout)
0dB
-6dB
-12dB
-18dB
-24dB
-30dB
-36dB
-42dB
-48dB
-54dB
-60dB
20°
0°
-20°
-40°
-60°
-80°
-100°
-120°
-140°
-160°
-180°
100Hz
1KHz
10KHz