Sinusoidal steady-state analysis

From our previous efforts with AC circuits, some patterns in the analysis started to appear.

1. In each case, the steady-state voltages or currents created in response to the sinusoidal source were themselves sinusoids operating at the source frequency, but having a distinct amplitudes and phase shifts relative to the source. The details of the amplitudes and phase shifts were determined completely by the amplitude of the source, the frequency, and the values of the components in the circuit.

2. There was also a transient part to each voltage or current. Depending on the order of the circuit, there were one or two transient terms. The details of the transient depended on the amplitude of the source, the frequency of the source, the values of the components in the circuit, any stored voltages or currents, and the exact timing of the when circuit change occurred (switch thrown or source activated).

3. The transients usually decay away within a couple of periods of the oscillation and had no effect on the long-term, steady-state behavior of the circuit.

4. The math used in solving the problem directly using diff. eq. techniques could be quite involved.

Sinusoidal steady-state analysis

In AC problems, we often are more interested in the steady-state behavior than in the transient part. This is not to say that transients are unimportant — in some situations, it may be absolutely vital to understand exactly what happens when a switched is flipped. But the steady-state behavior goes on "forever", and usually we need to understand that part before fussing with the details of the transients, which disappear in a relatively short time.

Therefore, we will simplify our analysis of AC circuits by ignoring the transient part of the solution. This has two consequences:

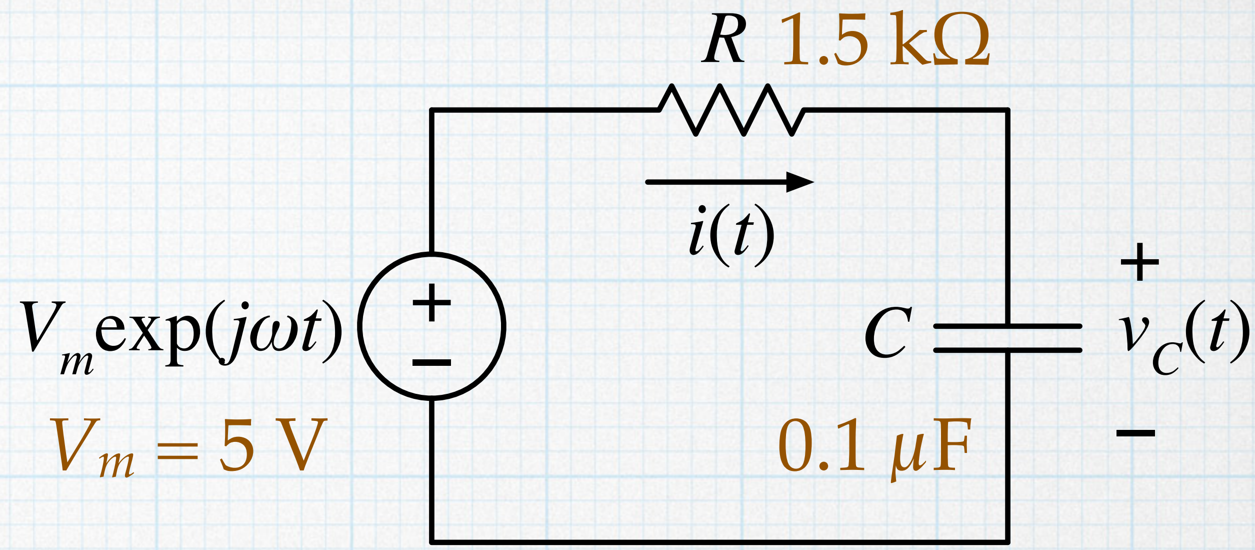
1. We will solve only the steady-state part of the problem, meaning that we will find $v_{ss}(t)$, but ignore $v_{tr}(t)$.
2. Since the initial conditions affect only the transient part of the complete solution, we can also ignore those, meaning that we will not worry about details like the exact timing of when a switch is moved or a source activated, and we won't care about initial voltages or currents. We assume that sinusoidal source has been on forever and will stay on forever.

This is simplification is known as **"sinusoidal steady-state analysis"**, but we will generally refer to it as **"AC analysis"**.

Complex sources (phasors)

Example 1: RC circuit

To test our new approach, we will rework the simple RC circuit from the previous set of notes. Everything is essentially same, expect the sinusoidal nature of the source is expressed using a complex sinusoid.



$\omega = 6660$ rad/s
($f = 1060$ Hz. $T = 0.943$ ms)

Since we decided to forego the transient analysis, we move directly to the steady-state solution.

$$\frac{dv_{ss}(t)}{dt} + \frac{v_{ss}(t)}{RC} = V_m \exp(\omega t)$$

As a trial function, we will use

$$v_{ss}(t) = A \exp\left(j\omega t\right)$$

where A is the complex voltage amplitude, as discussed previously.

Insert the trial function into the steady-state diff. eq.

$$\frac{d\left(Ae^{j\omega t}\right)}{dt} + \frac{Ae^{j\omega t}}{RC} = V_m e^{j\omega t}$$

$$j\omega\left(Ae^{j\omega t}\right) + \frac{1}{RC}\left(Ae^{j\omega t}\right) = V_m e^{j\omega t}$$

Divide by $e^{j\omega t}$

$$j\omega A + \frac{1}{RC}A = V_m$$

$$A = \frac{V_m}{1 + j\omega RC}$$

That was easy! Indeed, A is a complex amplitude, and we can express it in either real/imaginary or magnitude/phase form

$$A = \frac{V_m}{1 + (\omega RC)^2} - j\frac{(\omega RC)\,V_m}{1 + (\omega RC)^2} = \frac{V_m}{\sqrt{1 + (\omega RC)^2}}\exp\left(j\theta\right)$$

$$\theta = -\arctan\left(\omega RC\right)$$

These expressions might look somewhat familiar.

The steady-state solution, using complex sinusoids, is

$$v_{ss}(t) = A \exp\left(j\omega t\right)$$

With the magnitude/phase form of A,

$$v_{ss}(t) = \left[\frac{V_m}{\sqrt{1+(\omega RC)^2}} \exp\left(j\theta\right)\right] \exp\left(j\omega t\right)$$

$$v_{ss}(t) = \frac{V_m}{\sqrt{1+(\omega RC)^2}} \exp\left[j\left(\omega t + \theta\right)\right]$$

Plug in numbers:

$$v_{ss}(t) = \left(3.54\,\mathrm{V}\right) \exp\left[j\left(\omega t - 45^\circ\right)\right]$$

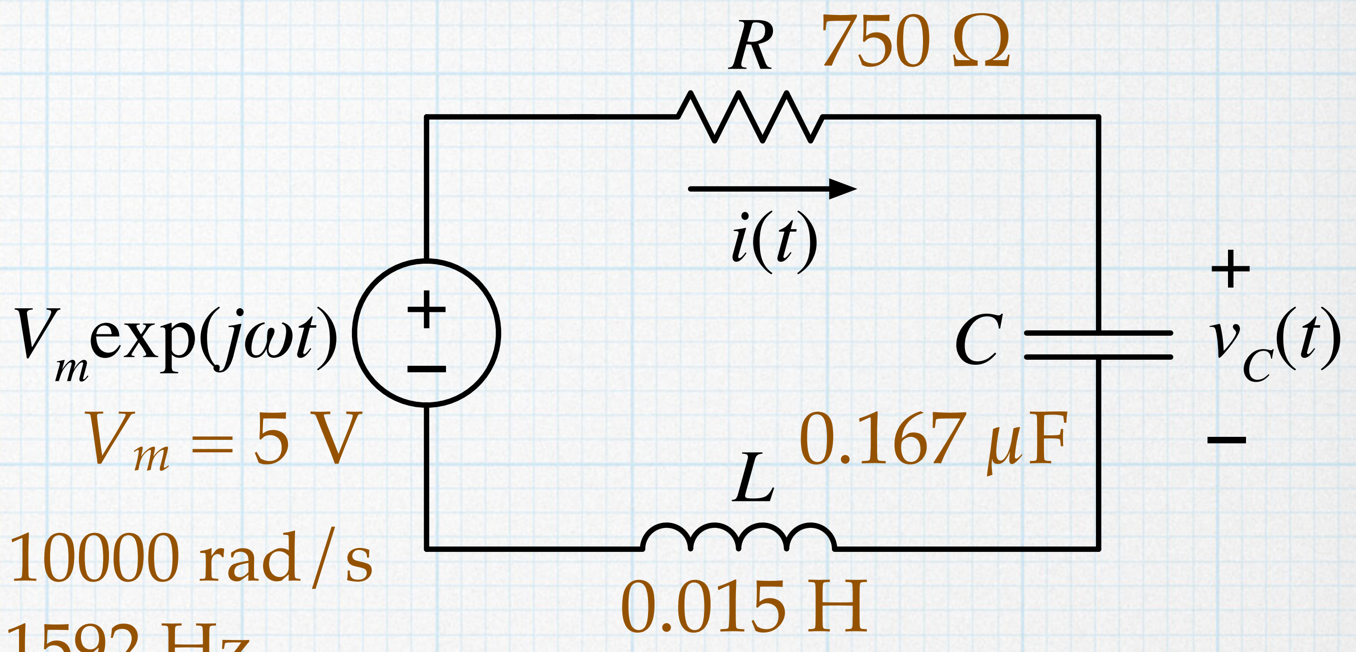
The real part of this complex quantity is

$$v_{ss}(t) = \left(3.54\,\mathrm{V}\right) \cos\left(\omega t - 45^\circ\right)$$

which is exactly what we found doing it “the hard way”.

A second example

Let's apply the complex approach to the *RLC* example done earlier. Again the sinusoidal is represented as complex exponential.



ω = 10000 rad/s
(f = 1592 Hz
T = 0.628 ms)

We move directly to the steady-state equation, replacing the original cosine function with the exponential sinusoid:

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_m}{LC}\exp\left(j\omega t\right)$$

Using $v_{ss}(t) = A\exp\left(j\omega t\right)$ worked well last time, so use it again.

$$\frac{d^2\left(Ae^{j\omega t}\right)}{dt^2} + \frac{R}{L}\frac{d\left(Ae^{j\omega t}\right)}{dt} + \frac{\left(Ae^{j\omega t}\right)}{LC} = \frac{\left(Ae^{j\omega t}\right)}{LC}$$

$$(j\omega)^2\left(Ae^{j\omega t}\right) + \frac{R}{L}(j\omega)\left(Ae^{j\omega t}\right) + \frac{\left(Ae^{j\omega t}\right)}{LC} = \frac{\left(V_m e^{j\omega t}\right)}{LC}$$

$$(j\omega)^2\left(Ae^{j\omega t}\right)+\frac{R}{L}(j\omega)\left(Ae^{j\omega t}\right)+\frac{\left(Ae^{j\omega t}\right)}{LC}=\frac{\left(V_m e^{j\omega t}\right)}{LC}$$

Divide by $e^{j\omega t}$, and collect terms:

$$\left[(j\omega)^2+\frac{R}{L}(j\omega)+\frac{1}{LC}\right]A=\frac{V_m}{LC}$$

and finish to find the complex voltage A:

$$A=\frac{V_m}{LC\left[(j\omega)^2+\frac{R}{L}(j\omega)+\frac{1}{LC}\right]}=\frac{V_m}{\left(1-\omega^2 LC\right)+j(\omega RC)}$$

This can be expressed in real/imaginary form

$$A=a+jb$$

$$a=\frac{V_m\left(1-\omega^2 LC\right)}{\left(1-\omega^2 LC\right)^2+(\omega RC)^2}=1.76\,\text{V}$$

$$b=-\frac{V_m(\omega RC)}{\left(1-\omega^2 LC\right)^2+(\omega RC)^2}=2.94\,\text{V}$$

Or magnitude and phase

$$M = \frac{V_m}{\sqrt{\left(1 - \omega^2 LC\right)^2 + (\omega RC)^2}} = 3.43\ \text{V}$$

$$A = M \exp\left(j\theta\right)$$

$$\theta = -\arctan\left(\frac{\omega RC}{1 - \omega^2 LC}\right) = -59^\circ$$

The complete steady-state solution in complex form is

$$v_{ss}(t) = A \exp\left(j\omega t\right)$$

$$= M \exp\left(j\theta\right) \exp\left(j\omega t\right)$$

$$= M \exp\left[j\left(\omega t + \theta\right)\right]$$

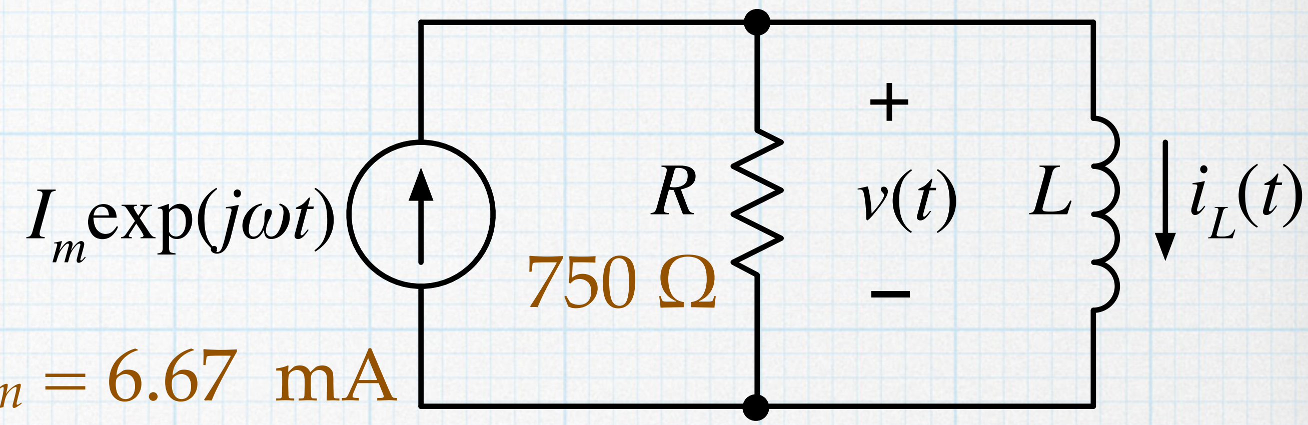
$$= (3.43\ \text{V}) \exp\left[j\left(\omega t - 59^\circ\right)\right]$$

The real part of this expression is:

$$vss(t) = (3.43\ \text{V}) \cos\left(\omega t - 59^\circ\right)$$ Identical.

Example 3: RL circuit

To finish off this section, we will use the complex approach with the *RL* circuit. The "hard way" details were left to a homework problem. But the complex approach is easy enough that we can just work it all here.



$I_m = 6.67$ mA

$\omega = 50000$ rad/s

$L = 15$ mH

($f = 7958$ Hz. $T = 0.126$ ms)

Recall that in the "hard way" notes, this example used $\sin(\omega t)$ dependence for the source. But in moving to the complex approach, we still use $\exp(j\omega t)$. Again, $\exp(j\omega t)$ has both $\cos(\omega t)$ and $\sin(\omega t)$ built into to it.

Start with a bit of circuit analysis:

$$v_R = v_L$$

$$R\left[I_m e^{j\omega t} - i_L(t)\right] = L\frac{di_L(t)}{dt}$$

$$\frac{di_L(t)}{dt} + \frac{R}{L}i_L(t) = \frac{R}{L}I_m e^{j\omega t}$$

Once again, jump straight to the steady-state equation,

$$\frac{di_{ss}(t)}{dt} + \frac{R}{L} i_{ss}(t) = \frac{R}{L} I_m e^{j\omega t}$$

Using $i_{ss}(t) = A \exp\left(j\omega t\right)$ as the trial function,

$$\frac{d\left(Ae^{j\omega t}\right)}{dt} + \frac{R}{L}\left(Ae^{j\omega t}\right) = \frac{R}{L}\left(I_m e^{j\omega t}\right)$$

$$j\omega\left(Ae^{j\omega t}\right) + \frac{R}{L}\left(Ae^{j\omega t}\right) = \frac{R}{L}\left(I_m e^{j\omega t}\right)$$

$$j\omega A + \frac{R}{L} A = \frac{R}{L} I_m$$

$$A = \frac{I_m}{1 + \frac{j\omega L}{R}}$$

As before, the complex current can be written in real/imaginary or magnitude/phase form.

$$A = a + jb$$

$$a = \frac{I_m}{1 + \left(\frac{\omega L}{R}\right)^2} = 3.33\,\text{mA}$$

$$b = -\frac{\left(\frac{\omega L}{R}\right) I_m}{1 + \left(\frac{\omega L}{R}\right)^2} = -\,3.33\,\text{mA}$$

$$A = M \exp\left(j\theta\right)$$

$$M = \frac{I_m}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} = 4.71\,\text{mA}$$

$$\theta = -\arctan\left(\frac{\omega L}{R}\right) = -\,45^\circ$$